# Point Optimal Panel Unit Root Tests with Serially Correlated Errors*†

Hyungsik Roger Moon
University of Maryland & University of Southern California

Benoit Perron
CIREQ, CIRANO, Université de Montréal

Peter C. B. Phillips
Yale University, University of Auckland,
Singapore Management University, University of Southampton

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**Abstract**

Generalizations of the point-optimal panel unit root tests of Moon, Perron, and Phillips (2007; MPP) are developed to cover cases of serially correlated errors. The resulting statistics involve two modifications relative to those in MPP: (i) the error variance is replaced by the long-run variance; (ii) centering of the statistic is adjusted to correct for second-order bias effects induced by the correlation between the error and lagged dependent variable.

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## 1 Introduction

There has been much recent interest in testing for the presence of stochastic trends in large panels (e.g., see Breitung and Pesaran (2008) and Breitung and

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Westerlund (2011)). A prototypical model consists of a deterministic trend component $d_{it}$ and an (unobserved) stochastic component component $y_{it}$ for some observable panel observations $z_{it}$ for individual $i = 1, ..., n$ in period $t = 1, ..., T$ satisfying

$$\begin{aligned} z_{it} &= d_{it} + y_{it}, \\ y_{it} &= \rho_i y_{it-1} + u_{it}, \end{aligned} \tag{1}$$

where $u_{it}$ is an error term that has zero mean and is stationary over time and $y_{i0} = 0$ for simplicity. Dynamic panel models with incidental trend components of this type arise in many applications in microeconometrics, multi-country growth studies, and international finance. Empirical interest often centers on the individual dynamics and whether there is commonality and persistence across individuals, i.e. that the autoregressive parameters $\rho_i$ are all unity, or whether such commonality occurs for certain subgroups of individuals.

Moon, Perron, and Phillips (2007, MPP thereafter) developed tests that are point optimal against a specific alternative hypothesis. MPP adopted a local-alternative setup, specifying the autoregressive parameter as lying in a local vicinity of unity whose width narrows as the sample size increases according to the form

$$\rho_i = 1 - \frac{\theta_i}{n^\kappa T} \text{ for some constant } \kappa > 0, \tag{2}$$

where $\theta_i$ is a sequence of iid random variables and $\kappa$ is a parameter defining the width of the vicinity as $n \to \infty$. The null hypothesis of interest is then

$$\mathbb{H}_0 : \ \theta_i = 0 \text{ a.s. } (i.e., \rho_i = 1) \text{ for all } i, \tag{3}$$

with the alternative

$$\mathbb{H}_1 : \theta_i \neq 0 \ \ (i.e., \ \rho_i \neq 1) \ \text{ for some } i's. \tag{4}$$

The MPP tests are point optimal in the sense of giving highest power against a specific set of $\theta_i's$. These tests were derived under the assumption that the error term $u_{it}$ is independent across individual units and over time.

Independence assumptions are not realistic in many empirical applications and the current work extends the MPP tests by allowing for serially correlated errors $u_{it}$. Section 6.4 (p. 436) of MPP briefly mentioned this extension. Here we provide explicit test statistics that have optimal asymptotic properties. The modified tests replace estimated variances of the errors in MPP with estimated long-run variances and adjust centering terms. Our main purpose is to provide the form of the modified tests and give their asymptotic properties so that they may be used in empirical work.

The paper is organized as follows. Section 2 shows how to construct the tests, gives results for cases with no fixed effects, fixed effects and incidental trends, and discusses implementation. Section 3 reports some simulations findings, Section 4 concludes, and the Appendix provides technical derivations and supporting lemmas.

# 2 Tests under Serial Correlation

Following MPP, the analysis below considers three deterministic trend cases: (i) no individual effects, that is, $d_{it} = 0$ and $z_{it} = y_{it}$; (ii) fixed effects, i.e., $d_{it} = b_{0i}$; and (iii) heterogenous or incidental linear trends, i.e., $d_{it} = b_{i0} + b_{i1}(t-1)$. In each case, we proceed in three steps. We first define the likelihood ratio (LR) statistic under Gaussianity, which is known to be optimal by the Neyman-Pearson lemma when the null and alternative hypotheses are simple. We then show that this statistic can be approximated by a simpler version with parameters that are consistently estimable. We finally derive the asymptotic distribution of this approximation (with appropriate recentering). In all three cases, this asymptotic distribution coincides with the one in MPP.

Our notation is similar to MPP. Denote by $Z$, $D$, $Y$, $Y_{-1}$, and $U$ the $(n \times T)$ observation matrices whose $(i,t)^{th}$ elements are $z_{it}$, $d_{it}$, $y_{it}$, $y_{it-1}$, and $u_{it}$, respectively. Define the $T-$ vectors $G_0 = (1, ...., 1)'$, $G_1 = (0, 1, ..., T-1)'$, set $G = (G_0, G_1) = (g_1, ..., g_T)'$, where $g_t = (1, t-1)'$. Define $\beta_0 = (b_{01}, ...., b_{0n})'$, $\beta_1 = (b_{11}, ...., b_{1n})'$, and $\beta = (\beta_0, \beta_1) = (b_1, ..., b_n)'$, where $b_i = (b_{0i}, b_{1i})'$. Let $\underline{Z}_i$, $\underline{Y}_i$, $\underline{Y}_{-1,i}$, and $\underline{U}_i$ denote the transpose of the $i^{th}$ row of $Z$, $Y$, $Y_{-1}$, and $U$, respectively. With this notation, the model has the matrix form

$$Z = D + Y, \quad Y = \rho Y_{-1} + U,$$

where $\rho = diag(\rho_1, ..., \rho_n)$.

Define $\sigma_i^2$, $\omega_i^2$, and $\lambda_i$ as the variance of $u_{it}$, the long-run variance of $u_{it}$ and the one-sided long-run variance of $u_{it}$, respectively, so that $\omega_i^2 = \sigma_i^2 + 2\lambda_i$. Let $\Sigma$, $\Omega$, and $\Lambda$ be the diagonal matrices with elements $\sigma_i^2$, $\omega_i^2$, and $\lambda_i$, respectively. Define $\Omega_{u,i} = E\left(\underline{U}_i \underline{U}_i'\right)$, the $(T \times T)$ covariance matrix of $\underline{U}_i$, and $\Omega_u = diag(\Omega_{u,1}, ..., \Omega_{u,n})$, the $(nT \times nT)$ covariance matrix of $vec(U')$. As in MPP, we assume that the errors $u_{it}$ are cross section independent over $i$.

We assume that the localizing coefficient $\theta_i$ in the local alternative (2) is a sequence of *iid* random variables with bounded support.[1] Let $\mu_{\theta,k} = E\left(\theta_i^k\right)$, $\rho_{c_i} = 1 - \frac{c_i}{n^\kappa T}$, and define the quasi-difference operator

$$\underset{(T \times T)}{\Delta_{c_i}} = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ -\rho_{c_i} & 1 & \ddots & \vdots & \vdots \\ 0 & \ddots & \ddots & 0 & 0 \\ \vdots & & -\rho_{c_i} & 1 & 0 \\ 0 & \dots & 0 & -\rho_{c_i} & 1 \end{bmatrix}.$$

Set $\mathbb{C} = diag(c_1, ..., c_n)$ and $\Delta_{\mathbb{C}} = diag(\Delta_{c_1}, ..., \Delta_{c_n})$.

[1] As mentioned in MPP, the assumption of a bounded support for $\theta_i$ is made for convenience, and could be relaxed at the cost of stronger moment conditions. It is also convenient to assume that the $\theta_i$ are identically distributed, and this assumption could be relaxed as long as cross sectional averages of the moments $\theta_i$ have well defined limits.

The quasi log-likelihood function of the panel $Z$ that we use in defining the likelihood ratio test statistic has the form

$$L_{nT}(\mathbb{C}, D, B) = -\frac{1}{2}(vec(Z' - D'))' \Delta_{\mathbb{C}}' B \Delta_{\mathbb{C}} (vec(Z' - D')),$$

for some weight matrix $B$.

Through the paper we will assume panel linear process errors with conditions similar to those in the literature (e.g., Phillips and Moon, 1999).

**Condition 1** *(a) Assume* $u_{it} = \sum_{j=0}^{\infty} c_{ij} v_{it-j}$*, where* $v_{it} \sim iid$ *with* $E(v_{it}) = 0$ *and* $E|v_{it}|^{8+\epsilon} < \infty$ *for some* $\epsilon > 0$*. Let* $c_j = \sup_i |c_{ij}|$. *(b) Assume* $\sum_{j=0}^{\infty} j^m c_j < \infty$ *for some* $m > 1$ *Let* $f_i(\lambda)$ *be the spectral density of* $u_{it}$*. Let* $\gamma_j(k) = \int_{-\pi}^{\pi} \exp(ik) f_j(\lambda) d\lambda$, $\gamma(k) = \sup_i |\gamma_i(k)|$, $\phi_j(k) = \int_{-\pi}^{\pi} \exp(ik)\left(4\pi^2 f_j(\lambda)\right)^{-1} d\lambda$, $\phi(k) = \sup_i |\phi_i(k)|$ *(c) Assume* $\gamma(k), \phi(k) \leq Mk^{-s}$ *and for* $s > 2$ *and some constant* $M$.

## 2.1 No Fixed Effect: $d_{it} = 0$

When $d_{it} = 0$, the model becomes

$$Z = Y, \; Y = \rho Y_{-1} + U.$$

Following MPP, in this case we consider local neighborhoods of unity that shrink at the rate of $\frac{1}{n^{1/2}T}$, so that the rate coefficient $\kappa = 1/2$, and one-sided alternatives in which the support of $\theta_i$ is a bounded interval $[0, M_\theta]$ for some $M_\theta \geq 0$ so that $\rho_i \leq 1$ under this alternative. In terms of the first moment of $\theta_i$ the hypotheses about $\rho_i$ are as follows:

$$\mathbb{H}_0 : \mu_{\theta,1} = 0, \tag{5}$$

and

$$\mathbb{H}_1 : \mu_{\theta,1} > 0. \tag{6}$$

Suppose that $u_{it}$ are Gaussian so that $vec(U') \sim N(0, \Omega_u)$ with known $\Omega_u$ and the initial conditions $y_{i0}$ are all zeros. By the Neyman-Pearson lemma, rejecting a small value of the log-likelihood ratio test statistic

$$-2L_{nT}\left(\mathbb{C}, 0, \Omega_u^{-1}\right) + 2L_{nT}\left(0, 0, \Omega_u^{-1}\right) \tag{7}$$

would be the uniformly most powerful test for the null $\rho_i = 1$ for $i = 1, ..., n$ against the simple alternative $\rho_i = 1 - \frac{c_i}{n^{1/2}T}$ for $i = 1, ..., n$. When the alternative is (4) with (6), this becomes a point optimal test.

In order to implement the optimal test statistic (7), one needs an estimate of the entire $(nT \times nT)$ covariance matrix $\Omega_u$. This is a huge high dimensional covariance estimation problem in a nonparametric set-up. The following theorem provides an approximation of the likelihood ratio test statistic in (7) with a statistic where the unknown nuisance parameters are consistently estimable.

**Theorem 2** *Assume Condition 1 with* $vec(U') \sim N(0, \Omega_u)$. *Assume that* $\frac{n}{T} \to 0$ *as* $n, T \to \infty$. *Then, for* $\rho_i = 1 - \frac{\theta_i}{n^{1/2}T}$, *we have*

$$\begin{aligned} & -2L_{nT}\left(\mathbb{C}, 0, \Omega_u^{-1}\right) + 2L_{nT}\left(0, 0, \Omega_u^{-1}\right) \\ = & -2L_{nT}\left(\mathbb{C}, 0, \Omega^{-1} \otimes I_T\right) + 2L_{nT}\left(0, 0, \Omega^{-1} \otimes I_T\right) - \frac{2}{n^{1/2}} l_n' \mathbb{C} \Omega^{-1} \Lambda l_n + o_p(1). \end{aligned}$$

Notice that the approximate likelihood ratio statistic

$$-2L_{nT}\left(\mathbb{C}, 0, \Omega^{-1} \otimes I_T\right) + 2L_{nT}\left(0, 0, \Omega^{-1} \otimes I_T\right) - \frac{2}{n^{1/2}} l_n' \mathbb{C} \Omega^{-1} \Lambda l_n \quad (8)$$

in Theorem 2 employs the Gaussian log-likelihood based on the long-run variance $\Omega \otimes I_T$ with an adjustment of the one-sided long run variance $\frac{2}{n^{1/2}} l_n' \mathbb{C} \Omega^{-1} \Lambda l_n$. The one-sided long run drift correction appears due to the correlation between the stationary error $u_{it}$ and the lagged dependent variable $z_{it-1} = y_{it-1}$. The main advantage of this formulation is that it involves quantities ($\Omega$ and $\Lambda$) that can be easily estimated consistently.

The test statistic we propose is to use the approximated log likelihood ratio (8) with appropriate centering. Define

$$V_{nT}(\mathbb{C}) = -2L_{nT}\left(\mathbb{C}, 0, \Omega^{-1} \otimes I_T\right) + 2L_{nT}\left(0, 0, \Omega^{-1} \otimes I_T\right) - \frac{1}{2}\mu_{c,2} - \frac{2}{\sqrt{n}} l_n' \mathbb{C} \Omega^{-1} \Lambda l_n,$$

$l_n = (1, ..., 1)$ is the sum vector and $\mu_{c,2} = E\left(c_i^2\right)$.

**Theorem 3** *Let Condition 1 hold and* $\frac{n}{T} \to 0$ *as* $n, T \to \infty$. *Then, under the local alternative* $\rho_i = 1 - \frac{\theta_i}{n^{1/2}T}$, *we have*

$$V_{nT}(\mathbb{C}) \Rightarrow N\left(-E(c_i \theta_i), 2\mu_{c,2}\right),$$

*where* $\mu_{c,2} = E\left(c_i^2\right)$.

**Remarks**

1. One can interpret the test statistic $V_{nT}(\mathbb{C})$ as an asymptotic version of the point optimal test for panel unit roots with possible serial correlation of unknown form in the error term.

2. Compared to the corresponding statistic in MPP which makes no allowance for serial correlation, there are two differences in $V_{nT}(\mathbb{C})$. First, as discussed in MPP, we use an estimate of the long-run covariance matrix $\Omega \otimes I_T$ instead of an estimate of the variance matrix $\Sigma \otimes I_T$ as the weight matrix. In addition, we recenter the statistic by subtracting the term $\frac{2}{\sqrt{n}} l_n' \mathbb{C} \Omega^{-1} \Lambda l_n$, which corrects for the correlation between the stationary error $u_{it}$ and the lagged dependent variable $z_{it-1} = y_{it-1}$. This term is not required for the test under temporal independence.

3. The limit distribution of $V_{nT}(\mathbb{C})$ is the same limit as in MPP (Theorem 6).

## 2.2 Time Invariant Fixed Effects: $d_{it} = b_{i0}$

In this section we consider the case where the incidental trends $d_{it} = b_{0i}$ are fixed over time. This corresponds to the standard fixed effects model. In this case, the model has matrix form

$$Z = \beta_0 G_0' + Y, \quad Y = \rho Y_{-1} + U.$$

As before**,** suppose that $vec\,(U') \sim N\,(0, \Omega_u)$ with known $\Omega_u$ and the initial conditions $y_{i0}$ are all zeros. Then, rejecting a small value of the test statistic,

$$-2\left[\min_{\beta_0} L_{nT}\left(\mathbb{C}, \beta_0 G_0', \Omega_u^{-1}\right) - \min_{\beta_0} L_{nT}\left(0, \beta_0 G_0', \Omega_u^{-1}\right)\right], \tag{9}$$

for the null $\rho_i = 1$ for $i = 1, ..., n$ and the alternative $\rho_i = 1 - \frac{c_i}{n^{1/2}T}$ for $i = 1, ..., n$, is known as the uniformly most powerful invariant test that is invariant with respect to the transformation $Z \to Z + \beta_0^* G_0'$ for arbitrary $\beta_0^*$. Against the alternative in $(4)$, this becomes a point optimal invariant test (e.g., Dufour and King(1991)).

As mentioned in the previous section, this statistic is difficult to implement due to the presence of $\Omega_u$, the full $(nT \times nT)$ covariance matrix of the error. This again motivates the use of an approximation.

**Theorem 4** *Assume Condition 1 with* $vec\,(U') \sim N\,(0, \Omega_u)$ *and let* $\frac{n}{T^{1/2}} \to 0$ *as* $n, T \to \infty$*. Then, for* $\rho_i = 1 - \frac{\theta_i}{n^{1/2}T}$*, we have*

$$\begin{aligned} & -2\left[\min_{\beta_0} L_{nT}\left(\mathbb{C}, \beta_0 G_0', \Omega_u^{-1}\right) - \min_{\beta_0} L_{nT}\left(0, \beta_0 G_0', \Omega_u^{-1}\right)\right] \\ = & -2\left[\min_{\beta_0} L_{nT}\left(\mathbb{C}, \beta_0 G_0', \Omega^{-1} \otimes I_T\right) - \min_{\beta_0} L_{nT}\left(0, \beta_0 G_0', \Omega^{-1} \otimes I_T\right)\right] \\ & -\frac{2}{n^{1/2}} l_n' \mathbb{C} \Omega^{-1} \Lambda l_n + o_p(1). \end{aligned}$$

**Remarks**

1. This approximation is derived under the stronger rate condition $\frac{n}{T^{1/2}} \to 0$ as $n, T \to \infty$ in place of the condition $\frac{n}{T} \to 0$ as $n, T \to \infty$ that is used without fixed effects.

2. The approximation involves the same correction for second-order bias as in the case without fixed effects.

Again, the test statistic we propose is the approximate log likelihood ratio (8) with appropriate centering. Define

$$\begin{aligned} V_{nT,fe1}(\mathbb{C}) = & -2\left[\min_{\beta_0} L_{nT}\left(\mathbb{C}, \beta_0 G_0', \Omega^{-1} \otimes I_T\right) - \min_{\beta_0} L_{nT}\left(0, \beta_0 G_0', \Omega^{-1} \otimes I_T\right)\right] \\ & -\frac{1}{2}\mu_{c,2} - \frac{2}{\sqrt{n}} l_n' \mathbb{C} \Omega^{-1} \Lambda l_n. \end{aligned}$$

**Theorem 5** *Assume Condition 1 holds and let* $\frac{n}{T} \to 0$ *as* $n, T \to \infty$*. Then, under the local alternative* $\rho_i = 1 - \frac{\theta_i}{n^{1/2}T}$*, we have*

$$V_{nT,fe1}(\mathbb{C}) \Rightarrow N\left(-E(c_i\theta_i), 2\mu_{c,2}\right),$$

*where* $\mu_{c,2} = E\left(c_i^2\right)$*.*

This asymptotic distribution is the same as without fixed effects and as in MPP (Theorem 9).

### 2.3 Incidental Trends: $d_{it} = b_{i0} + b_{i1}t$

Under heterogeneous linear trends we follow MPP and use local neighborhoods of unity that shrink at the slower rate of $\frac{1}{n^{1/4}T}$, so that the rate coefficient is $\kappa = 1/4$. The alternative may be two-sided, i.e. $\theta_i \sim iid$ with mean $\mu_\theta$ and variance $\sigma_\theta^2$, with a support that is a subset of a bounded interval $[-M_{l\theta}, M_{u\theta}]$, where $M_{l\theta}, M_{u\theta} \geq 0$. The slower rate of shrinkage in the local neighborhoods of unity is the result of the presence of heterogeneous trend effects in the panel. The presence of these *incidental trends* reduces discriminatory power in testing for the presence of common stochastic trends, so wider localizing intervals are needed to attain non trivial power functions.

Under these conditions, hypotheses (3) and (4) can be re-expressed as

$$\mathbb{H}_0 : \mu_{\theta,2} = 0, \tag{10}$$

and

$$\mathbb{H}_1 : \mu_{\theta,2} > 0. \tag{11}$$

Again, suppose that $vec(U') \sim N(0, \Omega_u)$ with known $\Omega_u$ and the initial conditions $y_{i0}$ are all zeros. Then, similar to the case of time invariant fixed effects, rejecting a small value of the test statistic,

$$-2\left[\min_\beta L_{nT}\left(\mathbb{C}, \beta G', \Omega_u^{-1}\right) - \min_\beta L_{nT}\left(0, \beta G', \Omega_u^{-1}\right)\right],$$

for the null $\rho_i = 1$ for $i = 1, ..., n$ and the alternative $\rho_i = 1 - \frac{c_i}{n^{1/4}T}$ for $i = 1, ..., n$, is known as the uniformly most powerful invariant test (with respect to the linear transformation $Z \to Z + \beta^* G'$ for arbitrary $\beta^*$), and against the alternative in (4), it becomes a point optimal invariant test. As before, we start by proving the validity of an approximation to this log-likelihood ratio.

**Theorem 6** *Assume Condition 1 with* $vec(U') \sim N(0, \Omega_u)$ *and let* $\frac{n}{T^{1/4}} \to 0$ *as* $n, T \to \infty$*. Then, for* $\rho_i = 1 - \frac{\theta_i}{n^{1/4}T}$*, we have*

$$\begin{aligned}
& -2\left[\min_\beta L_{nT}\left(\mathbb{C}, \beta G', \Omega_u^{-1}\right) - \min_\beta L_{nT}\left(0, \beta G', \Omega_u^{-1}\right)\right] \\
= \quad & -2\left[\min_\beta L_{nT}\left(\mathbb{C}, \beta G', \Omega^{-1} \otimes I_T\right) - \min_\beta L_{nT}\left(0, \beta G', \Omega^{-1} \otimes I_T\right)\right] \\
& -\frac{2}{n^{1/4}} l_n' \mathbb{C} \Omega^{-1} \Lambda l_n + o_p(1).
\end{aligned}$$

**Remarks**

1. This approximation is derived under the condition $\frac{n}{T^{1/4}} \to 0$ as $n, T \to \infty$, which is a stronger rate condition than that used for the intercepts case.

2. As before, the correction is due to the presence of a second-order bias term arising from the correlation between the lagged dependent variables and the error term.

Again we propose is to use the approximate log likelihood ratio with appropriate centering as a test statistic. Define

$$\begin{aligned} V_{nT,fe2}(\mathbb{C}) &= -2\left[\min_{\beta} L_{nT}\left(\mathbb{C}, \beta G', \Omega^{-1} \otimes I_T\right) - \min_{\beta} L_{nT}\left(0, \beta G', \Omega^{-1} \otimes I_T\right)\right] \\ &\quad + \frac{1}{n^{1/4}} l_n' \mathbb{C} \Omega^{-1} \Sigma l_n + \frac{1}{n^{1/2}}\left(l_n' \mathbb{C}^2 l_n\right) \omega_{p2T} + \frac{1}{n}\left(l_n' \mathbb{C}^4 l_n\right) \omega_{p4T}, \end{aligned}$$

where

$$\begin{aligned} \omega_{p2T} &= -\frac{1}{T}\sum_{t=1}^{T}\frac{t}{T} + \frac{2}{T}\sum_{t=1}^{T}\left(\frac{t}{T}\right)^2 - \frac{1}{3}, \\ \omega_{p4T} &= \frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\frac{t}{T}\frac{s}{T}\min\left(\frac{t}{T}, \frac{s}{T}\right) - \frac{2}{3}\frac{1}{T}\sum_{t=1}^{T}\left(\frac{t}{T}\right)^2 + \frac{1}{9}. \end{aligned}$$

**Theorem 7** *Assume Condition 1 holds and let* $\frac{n}{T} \to 0$ *as* $n, T \to \infty$. *Then, under the local alternative* $\rho_i = 1 - \frac{\theta_i}{n^{1/4}T}$, *we have*

$$V_{nT,fe2}(\mathbb{C}) \Rightarrow N\left(-\frac{1}{90}E\left(c_i^2\theta_i^2\right), \frac{1}{45}E\left(c_i^4\right)\right).$$

As before, $V_{fe2,nT}(\mathbb{C})$ reduces to the statistic from MPP when there is no serial correlation, and it has the same asymptotic distribution as in Theorem 13 of MPP.

## 2.4 Implementation of the tests

The test statistics $V_{nT}(\mathbb{C}), V_{nT,fe1}(\mathbb{C}),$ and $V_{nT,fe2}(\mathbb{C})$ depend on unknown parameters $\left\{\sigma_i^2\right\}, \left\{\omega_i^2\right\}$, and $\{\lambda_i\}$. Let $\hat{\sigma}_i^2$, $\hat{\omega}_i^2$, and $\hat{\lambda}_i$ be consistent estimators of $\sigma_i^2$, $\omega_i^2$, and $\lambda_i$, respectively. Similarly define the diagonal matrices of these elements as $\hat{\Sigma}$, $\hat{\Omega}$, and $\hat{\Lambda}$. To implement these tests, one may replace $\Sigma$, $\Omega$, and $\Lambda$ in $V_{nT}(\mathbb{C})$, $V_{nT,fe1}(\mathbb{C})$, and $V_{nT,fe2}(\mathbb{C})$ with $\hat{\Sigma}$, $\hat{\Omega}$, and $\hat{\Lambda}$, and we denote the test statistics as $\hat{V}_{nT}(\mathbb{C})$, $\hat{V}_{nT,fe1}(\mathbb{C})$, and $\hat{V}_{nT,fe2}(\mathbb{C})$. We assume the following regarding these estimators.

**Condition 8** $\sup_i E\left(\hat{\sigma}_i^2 - \sigma_i^2\right)^2 = o\left(\frac{1}{n}\right)$, $\sup_i E\left(\hat{\omega}_i^2 - \omega_i^2\right)^2 = o\left(\frac{1}{n}\right)$, *and* $\sup_i E\left(\hat{\lambda}_i^2 - \lambda_i^2\right)^2 = o\left(\frac{1}{n}\right)$ *under the local alternative.*

**Remarks**

1. An example of $\hat{\sigma}_i^2$ that satisfies Condition 8 is the time series sample variance of $\Delta z_{it}$ :

$$\hat{\sigma}_i^2 = \frac{1}{T-1}\sum_{t=2}^{T}\left(\Delta z_{it} - \left(\frac{1}{T-1}\sum_{t=2}^{T} z_{it}\right)\right)^2.$$

2. When kernel spectral density estimation is used for $\hat{\omega}_i^2$ and $\hat{\lambda}_i^2$ with bandwidth $h$, Condition 8 is satisfied if: (i) the kernel function $K(\cdot) : \mathbb{R} \to [0,1]$ is continuous at zero and all but a finite number of other points, satisfying $K(0) = 1,\ K(x) = K(-x),\ \int_{-\infty}^{\infty} K(x)^2 dx < M$, and $K_q = \lim_{x\to 0}\left[1 - K(x)/|x|^q\right] < \infty$ for some $0 < q \leq m$, where parameter $m$ is defined in Condition 1(b); and (ii) the bandwidth $h$ satisfies

$$\frac{nh}{T} + \frac{n}{h^{2q}} = o(1) \tag{12}$$

as $\frac{n}{T} \to 0$ and $h \to \infty$. (e.g., See Moon and Perron (2004)). If $\frac{n}{T} = o\left(T^{-a}\right)$ for some $0 < a < 1$ and $q \geq \frac{1}{2}\left(\frac{1-a}{a}\right)$, then the bandwidth condition (12) is satisfied if

$$T^{\frac{1}{2q}(1-a)} \lesssim h \lesssim T^a,$$

that is,

$$\frac{h}{T^a}, \frac{T^{\frac{1}{2q}(1-a)}}{h} = O(1).$$

**Theorem 9** *Under Conditions 1 and 8, as* $n, T \to \infty$ *with* $\frac{n}{T} \to \infty$, *we have* $\hat{V}_{nT}(\mathbb{C}) = V_{nT}(\mathbb{C}) + o_p(1)$, $\hat{V}_{nT,fe1}(\mathbb{C}) = V_{nT,fe1}(\mathbb{C}) + o_p(1)$, *and* $\hat{V}_{nT,fe2}(\mathbb{C}) = V_{nT,fe2}(\mathbb{C}) + o_p(1)$ *under the local alternative.*

# 3 Monte Carlo Simulations

This section reports the results of a small Monte Carlo experiment designed to assess the finite-sample properties of the tests presented above. For this purpose, we use the same DGP as MPP but employ either an AR1) or MA(1) process for the innovations so that the generating model has the following form:

$$\begin{aligned} z_{it} &= b_{0i} + b_{1i}t + y_{it}, \\ y_{it} &= \rho_i y_{it-1} + u_{it}, \end{aligned}$$

where the innovations follow either and AR(1) process:

$$\begin{aligned} u_{it} &= \gamma u_{it,t-1} + \varepsilon_{it} \\ \varepsilon_{it} &\sim iid\ N\left(0, \sigma_i^2\left(1-\gamma^2\right)\right) \end{aligned}$$

or an MA(1) process

$$\begin{aligned} u_{it} &= \varphi \varepsilon_{it-1} + \varepsilon_{it} \\ \varepsilon_{it} &\sim iid\ N\left(0, \sigma_i^2 \left(\frac{1}{1+\varphi^2}\right)\right). \end{aligned}$$

In both cases, we allow for heterogeneity and draw the idiosyncratic variance $\sigma_i^2$ from a uniform distribution, $\sigma_i^2 \sim U[0.5, 1.5]$. This variance is scaled such that the scale of $u_{it}$ is the same for all cases.

In both the incidental intercepts case $(b_{1i} = 0)$ and incidental trends case $(b_{1i} \neq 0)$, the parameters are drawn from $iidN(0,1)$. We focus the study on the size of the common point-optimal test with $c_i = 1$ for all $i$, as MPP advocated that choice. This implies that we set $\rho_i = 1$ for all $i$,which corresponds to $\theta_i = 0$ for all $i$ in our local-to-unity framework. We take three values for $n$ (10, 25, and 100) and two values of $T$ (100 and 250). All tests are conducted at the 5% significance level, and the number of replications is set at 10,000.

Estimation of the long-run variance and one-sided long run variance is critical to the performance of the test. In all cases, we estimate these quantities using a non-parametric estimator with quadratic spectral kernel and bandwidth selected in a data-based manner using the Andrews (1991) rule with prewhitening. Because estimation of long-run variances is difficult (especially in cases with negative moving average components), we also report results that use the unknown population values of $\omega_i' s$ (but still estimate $\sigma_i^2$ from the data).

The results are reported in table 1. The table is divided in two panels. The top panel reports the results for the incidental intercepts case, while the bottom panel shows the results for the incidental trends case. Each cell has two entries: the top entry reports rejections rates with estimated long-run variances, while the bottom entry reports the rejection rates with population long-run variance.

With estimated long-run variances, size is well-controlled in most cases. In fact, if anything, the test is generally conservative. The notable exception is in the presence of moving average components. In those cases, size is well controlled if we substitute the population long-run variances. Thus, the size distortions that are noticed can be attributed to the estimation of these parameters. It can also be noted that distortions get worse in the incidental trends case and as $N$ increases.

## 4 Conclusion

This paper develops generalizations of the point-optimal panel unit root tests of Moon, Perron, and Phillips (2007) to cover the case where the error term is serially correlated. The resulting statistics have two simple modifications relative to those in MPP. First, the variance of the errors is replaced by the long-run variance. Second, the centering of the statistic is adjusted to accommodate the second-order bias induced by the correlation between the error and lagged values of the dependent variable. Simulations show that these two adjustments lead to appropriately sized tests in most cases.

Another approach to dealing with serial dependence is to use the trend likelihood approach of Phillips (2011), which produces an alternative approximation for the implied likelihood based on a sieve approximation to the nonstationary components using an orthogonal series of trend basis functions. The idea of the trend likelihood approach is to project the observations on the sieve space and construct the likelihood and likelihood ratio statistic for the transformed observations using series based estimates of the long run variances (Phillips, 2005). This approach will be explored in later work.

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# Appendix

The appendix consists of three sections. In the first section we provide proofs of Theorems 2, 4, and 6 that approximate the Gaussian log-likelihood ratio statistic. In the second section we provide sketches of the proofs of the limit distribution results in Theorems 3, 5, and 7. In the third section, we provide a heuristic proof of Theorem 9. We only provide sketches of the proofs in the last two sections because the details are similar to those of the corresponding theorems in MPP and can be established with only minor modifications. Throughout the appendix, $M$ denotes a generic (finite) constant.

## 5 Proofs of the Approximations in Theorems 2, 4, and 6

### 5.1 Proof of Theorem 2

Here $\kappa = 1/2$. Assume Condition 1 and $\frac{n}{T} \to 0$. Since $\Delta \underline{Y}_i = -\frac{\theta_i}{n^\kappa T}\underline{Y}_{-1,i} + \underline{U}_i$, we can write

$$\begin{aligned}
& -2L_{nT}\left(\mathbb{C}, 0, \Omega_u^{-1}\right) + 2L_{nT}\left(0, 0, \Omega_u^{-1}\right) \\
= & \sum_{i=1}^{n}\left[\left(\Delta\underline{Y}_i + \frac{c_i}{n^\kappa T}\underline{Y}_{-1,i}\right)'\Omega_{u,i}^{-1}\left(\Delta\underline{Y}_i + \frac{c_i}{n^\kappa T}\underline{Y}_{-1,i}\right) - \left(\Delta\underline{Y}_i\right)'\Omega_{u,i}^{-1}\left(\Delta\underline{Y}_i\right)\right] \\
= & \frac{2}{n^\kappa T}\sum_{i=1}^{n} c_i \underline{Y}'_{-1,i}\Omega_{u,i}^{-1}\Delta\underline{Y}_i + \frac{1}{n^{2\kappa}T^2}\sum_{i=1}^{n} c_i^2 \underline{Y}'_{-1,i}\Omega_{u,i}^{-1}\underline{Y}_{-1,i}. \qquad (13)
\end{aligned}$$

Write

$$\begin{aligned}
\frac{2}{n^\kappa T}\sum_{i=1}^{n} c_i \underline{Y}'_{-1,i}\Omega_{u,i}^{-1}\Delta\underline{Y}_i &= \frac{1}{n^\kappa}\sum_{i=1}^{n} c_i\left[\frac{2}{T}\underline{Y}'_{-1,i}\Omega_{u,i}^{-1}\Delta\underline{Y}_i\right] \\
&= \frac{1}{n^\kappa}\sum_{i=1}^{n} c_i\left[\frac{2}{T}\frac{\underline{Y}'_{-1,i}\Delta\underline{Y}_i}{\omega_i^2} + \frac{\sigma_i^2}{\omega_i^2} - 1\right] + \frac{1}{n^\kappa}\sum_{i=1}^{n}\eta_{iT}
\end{aligned}$$

$$= \frac{2}{n^\kappa T}\sum_{i=1}^{n}\frac{c_i}{\omega_i^2}\underline{Y}'_{-1,i}\Delta\underline{Y}_i - \frac{2}{n^\kappa}\sum_{i=1}^{n} c_i\frac{\lambda_i}{\omega_i^2} + \frac{1}{n^\kappa}\sum_{i=1}^{n}\eta_{1iT}, \qquad (14)$$

where

$$\eta_{1iT} = 2c_i\left[\frac{1}{T}\underline{Y}'_{-1,i}\Omega_{u,i}^{-1}\Delta\underline{Y}_i - \frac{1}{T}\frac{\underline{Y}'_{-1,i}\Delta\underline{Y}_i}{\omega_i^2} + \frac{\lambda_i}{\omega_i^2}\right],$$

and

$$\frac{1}{n^{2\kappa}T^2}\sum_{i=1}^{n} c_i^2\underline{Y}'_{-1,i}\Omega_{u,i}^{-1}\underline{Y}_{-1,i} = \frac{1}{n^{2\kappa}T^2}\sum_{i=1}^{n}\frac{c_i^2}{\omega_i^2}\underline{Y}'_{-1,i}\underline{Y}_{-1,i} + \frac{1}{n^{2\kappa}}\sum_{i=1}^{n}\eta_{2iT}, \quad (15)$$

where

$$\eta_{2iT} = c_i^2 \left( \frac{1}{T^2} \underline{Y}'_{-1,i} \Omega_{u,i}^{-1} \underline{Y}_{-1,i} - \frac{1}{\omega_i^2 T^2} \underline{Y}'_{-1,i} \underline{Y}_{-1,i} \right).$$

In the following subsections we show that under Condition 1, as $\frac{n}{T} \to 0$,

$$\frac{1}{n^{1/4}} \sum_{i=1}^{n} \eta_{1iT} = o_p(1) \quad (16)$$

$$\frac{1}{n^{1/2}} \sum_{i=1}^{n} \eta_{2iT} = o_p(1). \quad (17)$$

Then, by $(13)-(17)$ with $\kappa = 1/2$, we deduce that

$$\begin{aligned} & -2L_{nT}\left(\mathbb{C}, 0, \Omega_u^{-1}\right) + 2L_{nT}\left(0, 0, \Omega_u^{-1}\right) \\ = & -\frac{2}{n^{1/2}T} \sum_{i=1}^{n} \frac{c_i}{\omega_i^2} \underline{Y}'_{-1,i} \Delta \underline{Y}_i + \frac{1}{nT^2} \sum_{i=1}^{n} \frac{c_i}{\omega_i^2} \underline{Y}'_{-1,i} \underline{Y}_{-1,i} - \frac{2}{n^{1/2}} \sum_{i=1}^{n} c_i \frac{\lambda_i}{\omega_i^2} + o_p(1) \\ = & -2L_{nT}\left(\mathbb{C}, 0, \Omega \otimes I_T\right) + 2L_{nT}\left(0, 0, \Omega \otimes I_T\right) - \frac{2}{\sqrt{n}} l_n' \mathbb{C} \Omega^{-1} \Lambda l_n + o_p(1), \end{aligned}$$

as required. ■

## 5.2 Proof of Theorem 4

Here $\kappa = 1/2$. Assume Condition 1 and $\frac{n}{T^{1/2}} \to 0$. By definition, we have

$$\begin{aligned} & -2\left[\min_{\beta_0} L_{nT}\left(\mathbb{C}, \beta_0 G_0', \Omega_u^{-1}\right) - \min_{\beta_0} L_{nT}\left(0, \beta_0 G_0', \Omega_u^{-1}\right)\right] \\ = & \sum_{i=1}^{n} \left[ (\Delta_{c_i} \underline{Y}_i)' \Omega_{u,i}^{-1} (\Delta_{c_i} \underline{Y}_i) - (\Delta \underline{Y}_i)' \Omega_{u,i}^{-1} (\Delta \underline{Y}_i) \right] \\ & - \sum_{i=1}^{n} \left[ \begin{array}{c} (\Delta_{c_i} \underline{Y}_i)' \Omega_{u,i}^{-1} (\Delta_{c_i} G_0) \left( (\Delta_{c_i} G_0)' \Omega_{u,i}^{-1} (\Delta_{c_i} G_0) \right)^{-1} (\Delta_{c_i} G_0)' \Omega_{u,i}^{-1} (\Delta_{c_i} \underline{Y}_i) \\ - (\Delta \underline{Y}_i)' \Omega_{u,i}^{-1} (\Delta G_0) \left( (\Delta G_0)' \Omega_{u,i}^{-1} (\Delta G_0) \right)^{-1} (\Delta G_0)' \Omega_{u,i}^{-1} (\Delta \underline{Y}_i) \end{array} \right]. \end{aligned}$$

By $(14), (15), (16)$, and $(17)$ we can approximate the first term as

$$\begin{aligned} & \sum_{i=1}^{n} \left[ (\Delta_{c_i} \underline{Y}_i)' \Omega_{u,i}^{-1} (\Delta_{c_i} \underline{Y}_i) - (\Delta \underline{Y}_i)' \Omega_{u,i}^{-1} (\Delta \underline{Y}_i) \right] \\ = & \sum_{i=1}^{n} \frac{1}{\omega_i^2} \left[ (\Delta_{c_i} \underline{Y}_i)' (\Delta_{c_i} \underline{Y}_i) - (\Delta \underline{Y}_i)' (\Delta \underline{Y}_i) \right] - \frac{2}{n^{1/2}} \sum_{i=1}^{n} c_i \frac{\lambda_i}{\omega_i^2} + o_p(1). \end{aligned}$$

Then, the required result for the theorem follows since

$$\begin{aligned}
& \sum_{i=1}^{n} \left(\Delta_{c_i}\underline{Y}_i\right)' \Omega_{u,i}^{-1} \left(\Delta_{c_i}G_0\right) \left(\left(\Delta_{c_i}G_0\right)' \Omega_{u,i}^{-1} \left(\Delta_{c_i}G_0\right)\right)^{-1} \left(\Delta_{c_i}G_0\right)' \Omega_{u,i}^{-1} \left(\Delta_{c_i}\underline{Y}_i\right) \\
& -\sum_{i=1}^{n} \frac{1}{\omega_i^2} \left(\Delta_{c_i}\underline{Y}_i\right)' \left(\Delta_{c_i}G_0\right) \left(\left(\Delta_{c_i}G_0\right)' \left(\Delta_{c_i}G_0\right)\right)^{-1} \left(\Delta_{c_i}G_0\right)' \left(\Delta_{c_i}\underline{Y}_i\right) \\
= \; & o_p(1) \qquad (18)
\end{aligned}$$

for any $c_i$ such that $\sup_i |c_i| < M$ for some constant $M$. The proof of (18) is available in the following subsection. ■

## 5.3 Proof of Theorem 6

Here $\kappa = 1/4$. Assume Condition 1 and $\frac{n}{T^{1/4}} \to 0$. By definition, we have

$$\begin{aligned}
& -2\left[\min_{\beta} L_{nT}\left(\mathbb{C}, \beta G', \Omega_u^{-1}\right) - \min_{\beta} L_{nT}\left(0, \beta G', \Omega_u^{-1}\right)\right] \\
= \; & \sum_{i=1}^{n} \left[\left(\Delta_{c_i}\underline{Y}_i\right)' \Omega_{u,i}^{-1} \left(\Delta_{c_i}\underline{Y}_i\right) - \left(\Delta\underline{Y}_i\right)' \Omega_{u,i}^{-1} \left(\Delta\underline{Y}_i\right)\right] \\
& -\sum_{i=1}^{n} \left[\begin{array}{c} \left(\Delta_{c_i}\underline{Y}_i\right)' \Omega_{u,i}^{-1} \left(\Delta_{c_i}G\right) \left(\left(\Delta_{c_i}G\right)' \Omega_{u,i}^{-1} \left(\Delta_{c_i}G\right)\right)^{-1} \left(\Delta_{c_i}G\right)' \Omega_{u,i}^{-1} \left(\Delta_{c_i}\underline{Y}_i\right) \\ -\left(\Delta\underline{Y}_i\right)' \Omega_{u,i}^{-1} \left(\Delta G\right) \left(\left(\Delta G\right)' \Omega_{u,i}^{-1} \left(\Delta G\right)\right)^{-1} \left(\Delta G\right)' \Omega_{u,i}^{-1} \left(\Delta\underline{Y}_i\right) \end{array}\right].
\end{aligned}$$

By $(14), (15), (16),$ and $(17)$, we can approximate the first term as

$$\begin{aligned}
& \sum_{i=1}^{n} \left[\left(\Delta_{c_i}\underline{Y}_i\right)' \Omega_{u,i}^{-1} \left(\Delta_{c_i}\underline{Y}_i\right) - \left(\Delta\underline{Y}_i\right)' \Omega_{u,i}^{-1} \left(\Delta\underline{Y}_i\right)\right] \\
= \; & \sum_{i=1}^{n} \frac{1}{\omega_i^2} \left[\left(\Delta_{c_i}\underline{Y}_i\right)' \left(\Delta_{c_i}\underline{Y}_i\right) - \left(\Delta\underline{Y}_i\right)' \left(\Delta\underline{Y}_i\right)\right] - \frac{2}{n^{1/4}} \sum_{i=1}^{n} c_i \frac{\lambda_i}{\omega_i^2} + o_p(1).
\end{aligned}$$

Also, in the following subsection we show that

$$\begin{aligned}
& \sum_{i=1}^{n} \left(\Delta_{c_i}\underline{Y}_i\right)' \Omega_{u,i}^{-1} \left(\Delta_{c_i}G\right) \left(\left(\Delta_{c_i}G\right)' \Omega_{u,i}^{-1} \left(\Delta_{c_i}G\right)\right)^{-1} \left(\Delta_{c_i}G\right)' \Omega_{u,i}^{-1} \left(\Delta_{c_i}\underline{Y}_i\right) \\
& -\sum_{i=1}^{n} \frac{1}{\omega_i^2} \left(\Delta_{c_i}\underline{Y}_i\right)' \left(\Delta_{c_i}G\right) \left(\left(\Delta_{c_i}G\right)' \left(\Delta_{c_i}G\right)\right)^{-1} \left(\Delta_{c_i}G\right)' \left(\Delta_{c_i}\underline{Y}_i\right) \\
= \; & o_p(1) \qquad (19)
\end{aligned}$$

for any $c_i$ such that $\sup_i |c_i| < M$ for some constant $M$. Then, we have

$$\begin{aligned}
& \sum_{i=1}^{n} \left[\begin{array}{c} \left(\Delta_{c_i}\underline{Y}_i\right)' \Omega_{u,i}^{-1} \left(\Delta_{c_i}G\right) \left(\left(\Delta_{c_i}G\right)' \Omega_{u,i}^{-1} \left(\Delta_{c_i}G\right)\right)^{-1} \left(\Delta_{c_i}G\right)' \Omega_{u,i}^{-1} \left(\Delta_{c_i}\underline{Y}_i\right) \\ -\left(\Delta\underline{Y}_i\right)' \Omega_{u,i}^{-1} \left(\Delta G\right) \left(\left(\Delta G\right)' \Omega_{u,i}^{-1} \left(\Delta G\right)\right)^{-1} \left(\Delta G\right)' \Omega_{u,i}^{-1} \left(\Delta\underline{Y}_i\right) \end{array}\right] \\
= \; & \sum_{i=1}^{n} \frac{1}{\omega_i^2} \left[\begin{array}{c} \left(\Delta_{c_i}\underline{Y}_i\right)' \left(\Delta_{c_i}G\right) \left(\left(\Delta_{c_i}G\right)' \left(\Delta_{c_i}G\right)\right)^{-1} \left(\Delta_{c_i}G\right)' \left(\Delta_{c_i}\underline{Y}_i\right) \\ -\left(\Delta\underline{Y}_i\right)' \left(\Delta G\right) \left(\left(\Delta G\right)' \left(\Delta G\right)\right)^{-1} \left(\Delta G\right)' \left(\Delta\underline{Y}_i\right) \end{array}\right] + o_p(1).
\end{aligned}$$

Combining these expressions gives the required result

$$\begin{aligned}
& -2\left[\min_{\beta} L_{nT}\left(\mathbb{C},\beta G',\Omega_u^{-1}\right)-\min_{\beta} L_{nT}\left(0,\beta G',\Omega_u^{-1}\right)\right] \\
= & -2\left[\min_{\beta} L_{nT}\left(\mathbb{C},\beta G',\Omega^{-1}\otimes I_T\right)-\min_{\beta} L_{nT}\left(0,\beta G',\Omega^{-1}\otimes I_T\right)\right] \\
& -\frac{2}{n^{1/4}} l_n'\mathbb{C}\Omega^{-1}\Lambda l_n+o_p(1). \blacksquare
\end{aligned}$$

## 5.4 Supplementary Results

### 5.4.1 A Useful Lemma

Before we start the proof of (16) and (17), we introduce a useful technical result. When $A$ is a matrix, we use three different norms, $\|A\|_o=\lambda_{\max}(A'A)^{1/2}$, where $\lambda_{\max}(\cdot)$ denotes the maximum eigenvalue, $\|A\|=tr(A'A)^{1/2}$, and $|A|=\sum_{i,j}|a_{ij}|$, where $a_{ij}$ is the $(i,j)^{th}$ element of $A$. It is well known that

$$\|A\|_o\leq\|A\|\leq|A|.$$

By definition, the covariance matrix of $\underline{U}_i$ is $\Omega_{u,i}=[\gamma_i(t-s)]_{t,s}$. Let $A_i$ be the $(T\times T)$ matrix whose $(t,s)$ element $a_{i,t,s}$ is $\rho_i^{t-s-1}$, if $t>s$, and zero, if $t\leq s$. Let

$$R_i=\omega_i\Omega_{u,i}^{-1/2}-\omega_i^{-1}\Omega_{u,i}^{1/2}. \tag{20}$$

**Lemma 10** *Assume Condition 1. Then* $\sup_i\frac{1}{T^{1/2}}\|R_iA_i\|<M$ *for some constant* $M$.

**Proof.** For the desired result, we show

$$\sup_i\frac{1}{T}\|R_iA_i\|^2\leq M.$$

By definition,

$$\begin{aligned}
\sup_i\frac{1}{T}\|R_iA_i\|^2 &= \sup_i\frac{1}{T}tr\left(A_i'R_i'R_iA_i\right)=\sup_i\frac{1}{T}tr\left(\omega_i^2A_i'\Omega_{u,i}^{-1}A_i+\omega_i^{-2}A_i'\Omega_{u,i}A_i-2A_i'A_i\right) \\
&\leq M\sup_i\left|\frac{1}{T}tr\left(A_i'\left(\Omega_{u,i}-\omega_i^2\right)A_i\right)\right|+M\sup_i\left|\frac{1}{T}tr\left(A_i'\left(\Omega_{u,i}^{-1}-\omega_i^{-2}\right)A_i\right)\right| \\
&= M(I+II),\ \text{say,}
\end{aligned}$$

where the inequality holds since $0<\inf_i\omega_i^2\leq\sup_i\omega_i^2<\infty$ under Condition 1 and by the triangle inequality.

First we show that $I = O(1)$. Define

$$
\begin{aligned}
a_{n,T,i}(k) &= \frac{1}{T}\sum_{t=1}^{T-k}\sum_{s=1}^{T} a_{i,t,s}a_{i,t+k,s} = \left(1-\frac{\theta_i}{n^{\kappa}T}\right)^{k}\sum_{s=1}^{T-k}\left(1-\frac{\theta_i}{n^{\kappa}T}\right)^{2(s-1)}\left(\frac{t-k-s}{T}\right) \\
a'_{n,T,i}(k) &= \sum_{s=1}^{T-k}\left(1-\frac{\theta_i}{n^{\kappa}T}\right)^{2(s-1)}\left(\frac{t-k-s}{T}\right) \\
a''_{n,T,i}(k) &= \sum_{s=1}^{T}\left(1-\frac{\theta_i}{n^{\kappa}T}\right)^{2(s-1)}\left(\frac{t-k-s}{T}\right) \\
a_{n,T,i}(0) &= \frac{1}{T}\left(\sum_{t=1}^{T} a_{i,t,s}\right)^{2} = \sum_{s=1}^{T}\left(1-\frac{\theta_i}{n^{\kappa}T}\right)^{2(s-1)}\left(\frac{t-s}{T}\right).
\end{aligned}
$$

By adding and subtracting the terms and the triangle inequality, we can bound

$$
\begin{aligned}
I &= \sup_i \left|\frac{1}{T} tr\left(A'_i\left(\Omega_{u,i}-\omega_i^2\right)A_i\right)\right| \\
&\leq I_1 + I_2 + I_3 + I_4,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= \sup_i 2\left|\sum_{k=1}^{T-1}\gamma_i(k)\left(a_{n,T,i}(k)-a'_{n,T,i}(k)\right)\right| \\
I_2 &= \sup_i 2\left|\sum_{k=1}^{T-1}\gamma_i(k)\left(a'_{n,T,i}(k)-a''_{n,T,i}(k)\right)\right| \\
I_3 &= \sup_i 2\left|\sum_{k=1}^{T-1}\gamma_i(k)\left(a''_{n,T,i}(k)-a_{n,T,i}(0)\right)\right| \\
I_4 &= \sup_i 2\left|a_{n,T,i}(0)\sum_{k=T}^{\infty}\gamma_i(k)\right|.
\end{aligned}
$$

For term $I_1$, note that since $\sup_i\left|\frac{1}{T}\sum_{s=1}^{T-k}\left(1-\frac{\theta_i}{n^{\kappa}T}\right)^{2(s-1)}\left(\frac{t-k-s}{T}\right)\right| < M$ and

$\sup_i |\theta_i| < M$, we have

$$
\begin{aligned}
& \left|\left(a_{n,T,i}(k) - a'_{n,T,i}(k)\right)\right| \\
= \; & T\left|\left(1-\frac{\theta_i}{n^\kappa T}\right)^k - 1\right|\left|\frac{1}{T}\sum_{s=1}^{T-k}\left(1-\frac{\theta_i}{n^\kappa T}\right)^{2(s-1)}\left(\frac{t-k-s}{T}\right)\right| \\
\leq \; & MT\left|\left(1-\frac{\theta_i}{n^\kappa T}\right)^k - 1\right| = MT\left|\sum_{j=1}^{k}\binom{k}{j}\left(\frac{-\theta_i}{n^\kappa T}\right)^j\right| \\
\leq \; & MT\sum_{j=1}^{k}\frac{1}{j!}\left(\frac{|\theta_i| k}{n^\kappa T}\right)^j = MT\frac{|\theta_i| k}{n^\kappa T}\sum_{j=1}^{k}\frac{1}{j!}\left(\frac{|\theta_i| k}{n^\kappa T}\right)^{j-1} \\
\leq \; & M\frac{k}{n^\kappa}\left(1+\sum_{j=1}^{\infty}\frac{1}{j!}\left(\frac{M}{n^\kappa}\right)^j\right) = M\frac{k}{n^\kappa}\exp\left(\frac{M}{n^\kappa}\right),
\end{aligned}
$$

where the second inequality uses $\binom{k}{j} \leq \frac{k^j}{j!}$, the last inequality uses $\sup_i |\theta_i| < M$ and $\frac{k}{T} \leq 1$, and the equality uses the Taylor representatoin of the exponential function. Then,

$$
I_1 \leq 2\sum_{k=1}^{T-1}\gamma(k)\sup_i\left|\left(a_{n,T,i}(k) - a'_{n,T,i}(k)\right)\right| \leq \frac{M}{n^\kappa}\left(\exp\left(\frac{M}{n^\kappa}\right)-1\right)\sum_{k=1}^{T-1}\gamma(k)k = o(1).
$$

For term $I_2$, notice that

$$
\begin{aligned}
& \left|\left(a'_{n,T,i}(k) - a''_{n,T,i}(k)\right)\right| \\
\leq \; & 2T\frac{1}{T}\sum_{s=T-k+1}^{T}\left(1+\frac{|\theta_i|}{n^\kappa T}\right)^{2(s-1)} \\
\leq \; & 2T\int_{1-\frac{k}{T}}^{1}\exp\left(\frac{2|\theta_i| r}{n^\kappa}\right)dr \\
= \; & \begin{cases} T\frac{n^\kappa}{|\theta_i|}\exp\left(\frac{2|\theta_i|}{n^\kappa}\right)\left[1-\exp\left(-\frac{2|\theta_i|}{n^\kappa}\frac{k}{T}\right)\right] & \text{for } \theta_i \neq 0 \\ 2k & \text{for } \theta_i = 0 \end{cases} \\
\leq \; & Mk,
\end{aligned}
$$

where the first inequality holds since $\theta_i \geq 0$ and $\left|\frac{t-k-s}{T}\right| \leq 2$ and the last inequality holds by the mean-value theorem and $\sup_i |\theta_i| < M$. Then,

$$
I_2 = \sup_i 2\left|\sum_{k=1}^{T-1}\gamma_i(k)\left(a'_{n,T,i}(k) - a''_{n,T,i}(k)\right)\right| \leq M\sum_{k=1}^{T-1}\gamma(k)k = O(1).
$$

For term $I_3$, we have

$$\begin{aligned} I_3 &= \sup_i 2 \left| \sum_{k=1}^{T-1} \gamma_i(k) \left( a''_{n,T,i}(k) - a_{n,T,i}(0) \right) \right| \\ &\leq 2 \sum_{k=1}^{T-1} \gamma(k)\, k \sup_i \left| \frac{1}{T} \sum_{s=1}^{T} \left( 1 - \frac{\theta_i}{n^\kappa T} \right)^{2(s-1)} \right| = O(1), \end{aligned}$$

the last line holds since $\sup_i \left| \frac{1}{T} \sum_{s=1}^{T} \left(1 - \frac{\theta_i}{n^\kappa T}\right)^{2(s-1)} \right| < M$. Finally, we have

$$\begin{aligned} I_4 &= \sup_i 2 \left| a_{n,T,i}(0) \sum_{k=T}^{\infty} \gamma_i(k) \right| \\ &\leq \sup_i \left| \frac{1}{T} \sum_{s=1}^{T} \left( 1 - \frac{\theta_i}{n^\kappa T} \right)^{2(s-1)} \left( \frac{t-s}{T} \right) \right| T \sum_{k=T}^{\infty} \gamma(k) \\ &\leq MT \sum_{k=T}^{\infty} k^{-s} \leq M \sum_{k=T}^{\infty} k^{-s+1} = o(1), \end{aligned}$$

where the second inequality holds since $\sup_i \left| \frac{1}{T} \sum_{s=1}^{T} \left(1 - \frac{\theta_i}{n^\kappa T}\right)^{2(s-1)} \left(\frac{t-s}{T}\right) \right| < M$. By combining terms $I_1 - I_4$, we have the required result

$$I = O(1).$$

The proof of $II = O(1)$ follows in a similar fashion and is omitted. ■

### 5.4.2 Proof of (17)

We prove the required result when $\frac{n}{T} \to 0$ and $\kappa = 1/4$. Since

$$\begin{aligned} E \left( \frac{1}{n^{1/2}} \sum_{i=1}^{n} \eta_{2iT} \right)^2 &= \left( \frac{1}{n^{1/2}} \sum_{i=1}^{n} E(\eta_{2iT}) \right)^2 + Var \left( \frac{1}{n^{1/2}} \sum_{i=1}^{n} \eta_{2iT} \right) \\ &\leq \left( n^{1/2} \sup_i |E(\eta_{2iT})| \right)^2 + \sup_i Var(\eta_{2iT}), \end{aligned}$$

the required result follows if we show

$$n^{1/2} \sup_i |E\eta_{2iT}| = o(1) \tag{21}$$

and

$$\sup_i Var(\eta_{2iT}) = o(1). \tag{22}$$

For $(21)$, we follow similar arguments used in proving $|E(S_1)| \to 0$ on page 831 (in the proof of Lemma A2) of ERS, and have for some constant $M$

$$n^{1/2} \sup_i |E\eta_{2iT}| \leq \left( \frac{n}{T} \right)^{1/2} M \sup_i \left( \frac{1}{\omega_i} \left\| \Omega_{u,i} \right\|_o \left\| \Omega_{u,i}^{-1} \right\|_o^{1/2} \frac{\|R_i A_i\|}{\sqrt{T}} \right) \leq M \left( \frac{n}{T} \right)^{1/2} = o(1),$$

where the second inequality holds since $0 < M_l \leq \inf_i f_i(\lambda) \leq \sup_i f_i(\lambda) \leq M_u < \infty$ and by Lemma 10, and the last inequality holds since $\frac{n}{T} \to 0$.

For (22), we also follow similar arguments to those used in proving $|Var(S_1)| \to 0$ on page 831 (in the proof of Lemma A2) of ERS, and have for some constant $M$

$$\sup_i Var(\eta_{2iT}) \leq \frac{1}{T} M \sup_i \left( \frac{1}{\omega_i^2} \|\Omega_{u,i}\|_o^2 \left\|\Omega_{u,i}^{-1}\right\|_o \frac{\|R_i A_i\|^2}{T} \right) \leq \frac{M}{T} = o(1). \blacksquare$$

### 5.4.3 Proof of (16)

We prove the required result when $\frac{n}{T} \to 0$ and $\kappa = 1/4$. By replacing $\Delta \underline{Y}_i$ in $\eta_{1iT}$ with $-\frac{\theta_i}{n^{1/4}T}\underline{Y}_{-1,i} + \underline{U}_i$, we can decompose $\eta_{1iT}$ as

$$\eta_{1iT} = \eta_{3iT} - \frac{1}{n^{1/4}}\eta_{4iT},$$

where

$$\begin{aligned} \eta_{3iT} &= 2c_i \left[ \frac{1}{T}\underline{Y}'_{-1,i}\Omega_{u,i}^{-1}\underline{U}_i - \frac{1}{T}\frac{\underline{Y}'_{-1,i}\underline{U}_i}{\omega_i^2} + \frac{\lambda_i}{\omega_i^2} \right] \\ \eta_{4iT} &= c_i\theta_i \left[ \frac{1}{T^2}\underline{Y}'_{-1,i}\Omega_{u,i}^{-1}\underline{Y}_{-1,i} - \frac{1}{T}\frac{\underline{Y}'_{-1,i}\underline{Y}_{-1,i}}{\omega_i^2} \right], \end{aligned}$$

and

$$\frac{1}{n^{1/4}}\sum_{i=1}^n \eta_{1iT} = \frac{1}{n^{1/4}}\sum_{i=1}^n \eta_{3iT} - \frac{1}{n^{1/2}}\sum_{i=1}^n \eta_{4iT}.$$

First, similar arguments to those in the proof of (17) lead to

$$\frac{1}{n^{1/2}}\sum_{i=1}^n \eta_{4iT} = o_p(1).$$

Then, the required result follows if

$$\frac{1}{n^{1/4}}\sum_{i=1}^n \eta_{3iT} = o_p(1),$$

which follows if

$$E\left(\frac{1}{n^{1/4}}\sum_{i=1}^n \eta_{3iT}\right)^2 = o(1).$$

Notice that

$$\begin{aligned} E\left(\frac{1}{n^{1/4}}\sum_{i=1}^n \eta_{3iT}\right)^2 &= \left(\frac{1}{n^{1/4}}\sum_{i=1}^n E(\eta_{3iT})\right)^2 + Var\left(\frac{1}{n^{1/4}}\sum_{i=1}^n \eta_{3iT}\right) \\ &\leq \left(n^{3/4}\sup_i |E(\eta_{3iT})|\right)^2 + n^{1/2}\sup_i Var(\eta_{3iT}), \end{aligned}$$

By similar arguments to those used for the proof of $\sup_i Var(\eta_{2iT}) = O\left(\frac{1}{T}\right),$ we can show that

$$n^{1/2}\sup_i Var(\eta_{3iT}) = n^{1/2}O\left(\frac{1}{T}\right) = o(1).$$

For $n^{3/4}\sup_i |E(\eta_{3iT})| = o(1)$, we show

$$\sup_i |E(\eta_{3iT})| = O\left(\frac{1}{T}\right). \tag{23}$$

Since $\frac{n}{T} \to 0$, the desired result follows. Since $\underline{Y}_{-1,i} = A_i\underline{U}_i$, we have

$$\eta_{3iT} = 2c_i\left[\frac{1}{T}A_i'\underline{U}_i'\Omega_{u,i}^{-1}\underline{U}_i - \frac{1}{T}\frac{A_i'\underline{U}_i'\underline{U}_i}{\omega_i^2} + \frac{\lambda_i}{\omega_i^2}\right].$$

Since $tr(A_i) = 0$, we have

$$\begin{aligned}
E(\eta_{3iT}) &= 2c_i\frac{1}{T}\left[tr(A_i) - \frac{1}{\omega_i^2}tr(\Omega_{u,i}A_i)\right] + 2c_i\frac{\lambda_i}{\omega_i^2} \\
&= \frac{-2c_i}{\omega_i^2}\left[\sum_{k=1}^{T}\gamma_i(k)\left(1-\frac{k}{T}\right)\rho_i^{k-1} - \sum_{k=1}^{\infty}\gamma_i(k)\right] \\
&= \frac{-2c_i}{\omega_i^2}\left[\sum_{k=1}^{T}\gamma_i(k)\left(\rho_i^{k-1}-1\right) - \frac{1}{T}\sum_{k=1}^{T}k\gamma_i(k)\rho_i^{k-1} - \sum_{k=T+1}^{\infty}\gamma_i(k)\right] \\
&= I + II + III, \text{ say.}
\end{aligned}$$

For term $I$, we can bound

$$\begin{aligned}
0 &\leq \left|1-\rho_i^{k-1}\right| = \left|1-\left(1-\frac{\theta_i}{n^\kappa T}\right)^{k-1}\right| \leq \sum_{j=1}^{k-1}\binom{k-1}{j}\left(\frac{|\theta_i|}{n^\kappa T}\right)^j \\
&\leq \sum_{j=1}^{k-1}\frac{1}{j!}\left(\frac{|\theta_i|(k-1)}{n^\kappa T}\right)^j = \frac{|\theta_i|(k-1)}{n^\kappa T}\sum_{j=1}^{k-1}\frac{1}{j!}\left(\frac{|\theta_i|(k-1)}{n^\kappa T}\right)^{j-1} \\
&\leq \frac{M(k-1)}{n^\kappa T}\sum_{j=1}^{k-1}\frac{1}{j!}\left(\frac{M}{n^\kappa}\right)^{j-1} \leq \frac{M(k-1)}{n^\kappa T}\left(1+\sum_{j=1}^{\infty}\frac{1}{j!}\left(\frac{M}{n^\kappa}\right)^j\right) \\
&\leq \frac{M(k-1)}{n^\kappa T}\left(\exp\left(\frac{M}{n^\kappa}\right)\right).
\end{aligned}$$

Then, for some constant $M > 0$, we can bound

$$|I| = \sup_i\frac{2c_i}{\omega_i^2}\sum_{k=1}^{T}|\gamma_i(k)|\left|\left(1-\rho_i^{k-1}\right)\right| \leq \frac{M}{n^\kappa T}\left(\exp\left(\frac{M}{n^\kappa}\right)\right)\sum_{k=1}^{T}\gamma(k)k = o\left(\frac{1}{T}\right),$$

as required. Next,

$$|II| \leq \frac{M}{T} \sum_{k=1}^{T} k\gamma(k) = O\left(\frac{1}{T}\right),$$

and

$$\begin{aligned} |III| &\leq M \sum_{k=T+1}^{\infty} \gamma(k) \leq M \sum_{k=T+1}^{\infty} k^{-s} \text{ by Condition (iii)} \\ &\leq M(T+1)^{-s+1} = o\left(\frac{1}{T}\right) \text{ since } s > 2, \end{aligned}$$

as required. ■

#### 5.4.4 More Preliminary Results

In this section $\kappa = 1/4$ and we assume Condition 1. Define $\Phi_i$ to the $(T \times T)$ matrix whose $(r,s)^{th}$ element is $\phi_i(r-s)$, where $\phi_i(k)$ is defined in Condition 1.

Define $\tilde{G} = \left[\tilde{G}_0, \tilde{G}_1\right] = [G_0, G_1]\left(diag\left(\sqrt{T}, 1\right)\right)$. Direct calculations show that

$$\begin{aligned} \Delta_{c_i}\tilde{G}_0 &= \left(T^{1/2}, \frac{c_i}{n^{1/4}T^{1/2}}, ..., \frac{c_i}{n^{1/4}T^{1/2}}\right)', \\ \Delta_{c_i}\tilde{G}_1 &= \left(0, 1+\frac{c_i}{n^{1/4}}\frac{1}{T}, ..., 1+\frac{c_i}{n^{1/4}}\frac{t-1}{T}, ..., 1+\frac{c_i}{n^{1/4}}\frac{T-1}{T}\right)', \\ \frac{1}{T}\left(\Delta_{c_i}\tilde{G}\right)'\left(\Delta_{c_i}\tilde{G}\right) &= \begin{pmatrix} 1+\frac{c_i^2}{n^{1/2}T}\frac{T-1}{T} & \frac{1}{T^{1/2}}\left(\frac{c_i}{n^{1/4}}+\frac{c_i^2}{n^{1/2}}\frac{1}{T}\sum_{t=2}^{T}\frac{t-1}{T}\right) \\ \frac{1}{T^{1/2}}\left(\frac{c_i}{n^{1/4}}+\frac{c_i^2}{n^{1/2}}\frac{1}{T}\sum_{t=2}^{T}\frac{t-1}{T}\right) & \frac{1}{T}\sum_{t=2}^{T}\left(1+\frac{c_i}{n^{1/4}}\frac{t-1}{T}\right)^2 \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{T^{1/2}}\left(\Delta_{c_i}\tilde{G}\right)'\left(\Delta_{c_i}\underline{Y}_i\right) \\ = &\begin{pmatrix} y_{i1}+\frac{c_i}{n^{1/4}T^{1/2}}\frac{1}{T^{1/2}}\left(y_{iT}-y_{i1}\right)+\frac{c_i^2}{n^{1/2}T^{1/2}}\frac{1}{T^{3/2}}\sum_{t=2}^{T}y_{it-1} \\ \frac{1}{T^{1/2}}\left(y_{iT}-y_{i1}\right)+\frac{c_i}{n^{1/4}T^{1/2}}\left(y_{iT}-\frac{1}{T}\left(y_{iT}+y_{i0}\right)\right)+\frac{c_i^2}{n^{1/2}}\frac{1}{T^{3/2}}\sum_{t=2}^{T}\frac{t-1}{T}y_{it-1} \end{pmatrix} \quad (24) \end{aligned}$$

Define

$$b_{n,T,i}^{jl}(k) = \frac{1}{T}\sum_{t=2}^{T-k}\left[\left(\Delta_{c_i}\tilde{G}_j\right)_t\left(\Delta_{c_i}\tilde{G}_l\right)_{t+k}+\left(\Delta_{c_i}\tilde{G}_l\right)_t\left(\Delta_{c_i}\tilde{G}_j\right)_{t+k}\right],$$

where $(x)_t$ is the $t^{th}$ element of the vector $x$ and $j,l = 0,1$.

**Lemma 11** *(a)* $\sup_i\left|b_{n,T,i}^{01}(k)\right| \leq \frac{M}{T^{1/2}}$ *for all* $k$. *(b)* $\sup_i\left|b_{n,T,i}^{11}(k) - b_{n,T,i}^{11}(0)\right| \leq M\frac{k}{T}$ *for some finite constant* $M$.

**Proof.** Part (a): By definition, for $k=0$,

$$\sup_i \left| b^{01}_{n,T,i}(0) \right| = 2\sup_i \left| \frac{1}{T}\left(\Delta_{c_i}\tilde{G}_0\right)'\left(\Delta_{c_i}\tilde{G}_1\right)\right| = \frac{2}{T^{1/2}}\sup_i \left| \frac{c_i}{n^{1/4}} + \frac{c_i^2}{n^{1/2}}\frac{1}{T}\sum_{t=2}^{T}\frac{t-1}{T}\right| \leq \frac{M}{n^{1/4}T^{1/2}}.$$

For $k \geq 1$, we have

$$\sup_i \left| b^{01}_{n,T,i}(k) \right| = \sup_i \left| \frac{1}{T}\sum_{t=2}^{T-k}\left[\left(\Delta_{c_i}\tilde{G}_0\right)_t\left(\Delta_{c_i}\tilde{G}_1\right)_{t+k} + \left(\Delta_{c_i}\tilde{G}_1\right)_t\left(\Delta_{c_i}\tilde{G}_0\right)_{t+k}\right]\right| \leq \frac{M}{T^{1/2}},$$

as required. ■

Part (b): By definition,

$$\begin{aligned}
& \sup_i \left| b^{11}_{n,T,i}(k) - b^{11}_{n,T,i}(0) \right| \\
\leq\ & 2\frac{1}{T}\sum_{t=2}^{T-k}\left|\left(1+\frac{c_i}{n^{1/4}T}\frac{t-1}{T}\right)\left(1+\frac{c_i}{n^{1/4}T}\frac{t+k-1}{T}\right) - \left(1+\frac{c_i}{n^{1/4}T}\frac{t-1}{T}\right)^2\right| \\
& +2\frac{1}{T}\sum_{t=T-k+1}^{T}\left(1+\frac{c_i}{n^{1/4}T}\frac{t-1}{T}\right)^2 \\
\leq\ & \frac{M}{n^{1/4}}\frac{k}{T} + M\frac{k}{T},
\end{aligned}$$

as required. ■

**Lemma 12** *(a) Suppose that $x_i$ and $z_i$ are $T-$ vectors such that $\sup_{i,t}|z_{it}|$ is bounded, where $z_{it}$ is the $t^{th}$ element of $z_i$. Then, $\sup_i \left|\frac{1}{T}x_i'\left(\Omega_{u,i}^{-1} - \Phi_i\right)z_i\right| = O\left(\frac{\sup_i\|x_i\|}{T}\right)$. (b) $\sup_i \frac{1}{T}\left\|R_i\left(\Delta_{c_i}\tilde{G}_1\right)\right\|^2 = O\left(\frac{1}{T^{1/2}}\right)$, where $R_i$ is defined in* (20).

**Proof.** Part (a): The proof is similar to that of Lemma A1 of ERS and is omitted. ■

Part (b): We replace $A_i$ in the proof of Lemma 10 with $\left(\Delta_{c_i}\tilde{G}_1\right)$. Then, the required result follows if we show

$$\text{(b1):} \qquad \sup_i \left|\frac{1}{T}\left(\Delta_{c_i}\tilde{G}_1\right)'\left(\Omega_{u,i} - \omega_i^2\right)\left(\Delta_{c_i}\tilde{G}_1\right)\right| = O\left(\frac{1}{T}\right)$$

$$\text{(b2):} \qquad \sup_i \left|\frac{1}{T}\left(\Delta_{c_i}\tilde{G}_1\right)'\left(\Omega_{u,i}^{-1} - \omega_i^{-2}\right)\left(\Delta_{c_i}\tilde{G}_1\right)\right| = O\left(\frac{1}{T^{1/2}}\right).$$

For Part (b1), by definition, we have

$$\begin{aligned}
&\sup_i \left| \frac{1}{T} \left(\Delta_{c_i}\tilde{G}_1\right)' \left(\Omega_{u,i} - \omega_i^2\right) \left(\Delta_{c_i}\tilde{G}_1\right)' \right| \\
= & \sup_i \left| \sum_{k=1}^{T-1} \gamma_i(k) \left(b_{n,T,i}^{11}(k) - b_{n,T,i}^{11}(0)\right) - \left(\sum_{k=T}^{\infty} \gamma_i(k)\right) b_{n,T,i}^{11}(0) \right| \\
\leq & \sup_i \left| \sum_{k=1}^{T-1} \gamma_i(k) \left(b_{n,T,i}^{11}(k) - b_{n,T,i}^{11}(0)\right) \right| + \sup_i \left| \left(\sum_{k=T}^{\infty} \gamma_i(k)\right) b_{n,T,i}^{11}(0) \right|.
\end{aligned}$$

By by Lemma 11(b), the first term is bounded by

$$\sum_{k=1}^{T-1} \gamma(k) \sup_i \left| b_{n,T,i}^{11}(k) - b_{n,T,i}^{11}(0) \right| \leq M \frac{1}{T} \sum_{k=1}^{T-1} \gamma(k) k = O\left(\frac{1}{T}\right),$$

as required. Under Condition 1(iii), the second term is bounded by

$$\left(\sum_{k=T}^{\infty} k^{-s}\right) \sup_i \left| b_{n,T,i}^{11}(0) \right| \leq o\left(\frac{1}{T}\right),$$

as required.

For Part (b2), we have

$$\begin{aligned}
&\sup_i \left| \frac{1}{T} \left(\Delta_{c_i}\tilde{G}_1\right)' \left(\Omega_{u,i}^{-1} - \omega_i^{-2}\right) \left(\Delta_{c_i}\tilde{G}_1\right) \right| \\
\leq & \sup_i \left| \frac{1}{T} \left(\Delta_{c_i}\tilde{G}_1\right)' \left(\Omega_{u,i}^{-1} - \Phi_i\right) \left(\Delta_{c_i}\tilde{G}_1\right) \right| + \sup_i \left| \frac{1}{T} \left(\Delta_{c_i}\tilde{G}_1\right)' \left(\Phi_i - \omega_i^{-2}\right) \left(\Delta_{c_i}\tilde{G}_1\right) \right|.
\end{aligned}$$

By Part (a), we have

$$\sup_i \left| \frac{1}{T} \left(\Delta_{c_i}\tilde{G}_1\right)' \left(\Omega_{u,i}^{-1} - \Phi_i\right) \left(\Delta_{c_i}\tilde{G}_1\right) \right| \leq O\left(\sup_i \frac{\left\| \Delta_{c_i}\tilde{G}_1 \right\|}{T}\right) = O\left(\frac{1}{T^{1/2}}\right).$$

Using similar argument used in the proof of Part (b1), we can bound the second term by

$$\sup_i \left| \frac{1}{T} \left(\Delta_{c_i}\tilde{G}_1\right)' \left(\Phi_i - \omega_i^{-2}\right) \left(\Delta_{c_i}\tilde{G}_1\right) \right| \leq O\left(\frac{1}{T}\right).$$

Combining these, we have the required result for Part (b2). ■

For $\mathbb{C} = diag(c_1, ..., c_n)$, we define

$$\begin{aligned}
A_{iT}(\mathbb{C}) &= \frac{1}{T^{1/2}} \left(\Delta_{c_i}\tilde{G}\right)' \Omega_{u,i}^{-1} \left(\Delta_{c_i}\underline{Y}_i\right), \ A_{iT}^*(\mathbb{C}) = \frac{1}{\omega_i^2} \frac{1}{T^{1/2}} \left(\Delta_{c_i}\tilde{G}\right)' \left(\Delta_{c_i}\underline{Y}_i\right) \\
B_{iT}(\mathbb{C}) &= \frac{1}{T} \left(\Delta_{c_i}\tilde{G}\right)' \Omega_{u,i}^{-1} \left(\Delta_{c_i}\tilde{G}\right), \ B_{iT}^*(\mathbb{C}) = \frac{1}{\omega_i^2} \frac{1}{T} \left(\Delta_{c_i}G\right)' \left(\Delta_{c_i}G\right), \\
\tilde{B}_{iT}(\mathbb{C}) &= diag\left(B_{11,iT}(\mathbb{C}), B_{22,iT}(\mathbb{C})\right), \ \tilde{B}_{iT}^*(\mathbb{C}) = diag\left(B_{11,iT}^*(\mathbb{C}), B_{22,iT}^*(\mathbb{C})\right)
\end{aligned}$$

and we will define $B_{kl,iT}(\mathbb{C})$ to the $(k,l)^{th}$ element of $B_{iT}(\mathbb{C})$ and $A_{k,iT}(\mathbb{C})$ to be the $k^{th}$ element of $A_{k,iT}(\mathbb{C})$, where $k,l=1,2$. Similarly we define $A^*_{k,iT}(\mathbb{C})$ and $B^*_{kl,iT}(\mathbb{C})$.

**Lemma 13** *Under Conditions 1, the following hold.*
*(a)* $\sup_i |B_{12,iT}(\mathbb{C})|, \sup_i |B^*_{12,iT}(\mathbb{C})| = O\left(\frac{1}{T^{1/2}}\right)$.
*(b)* $\sup_i \left\|B_{iT}(\mathbb{C}) - \tilde{B}_{iT}(\mathbb{C})\right\|, \sup_i \left\|B^*_{iT}(\mathbb{C}) - \tilde{B}^*_{iT}(\mathbb{C})\right\| = O\left(\frac{1}{T^{1/2}}\right)$.
*(c)* $\sup_i \left\|B_{iT}(\mathbb{C})^{-1}\right\|, \sup_i \left\|\tilde{B}_{iT}(\mathbb{C})^{-1}\right\| \leq M$.
*(d)* $\sup_i E\left\|A_{iT}(\mathbb{C})\right\|^2, \sup_i E\left\|A^*_{iT}(\mathbb{C})\right\|^2 \leq M$.
*(e)* $\sup_i |B_{11,iT}(\mathbb{C}) - B_{11,iT}(0)| = O\left(\frac{1}{n^{1/4}T^{1/2}}\right)$.
*(f)* $\sup_i |B_{22,iT}(\mathbb{C}) - B^*_{22,iT}(\mathbb{C})| = O\left(\frac{1}{T^{1/2}}\right)$.
*(g)* $\sup_i E\left|A_{2,iT}(\mathbb{C}) - A^*_{2,iT}(\mathbb{C})\right|^2 = O\left(\frac{1}{T^{1/2}}\right)$.

**Proof.** Part (a): A direct calculation shows that $\sup_i \left|B^*_{12,iT}(\mathbb{C})\right| = O\left(\frac{1}{T^{1/2}}\right)$. We bound $\sup_i |B_{12,iT}(\mathbb{C})|$ by

$$\begin{aligned}\sup_i |B_{12,iT}(\mathbb{C})| &= \sup_i \left|\frac{1}{T}\left(\Delta_{c_i}\tilde{G}_0\right)' \Omega^{-1}_{u,i}\left(\Delta_{c_i}\tilde{G}_1\right)\right| \\ &\leq \sup_i \left|\frac{1}{T}\left(\Delta_{c_i}\tilde{G}_0\right)'\left(\Omega^{-1}_{u,i} - \Phi_i\right)\left(\Delta_{c_i}\tilde{G}_1\right)\right| + \sup_i \left|\frac{1}{T}\left(\Delta_{c_i}\tilde{G}_0\right)'\left(\Phi_i - \frac{1}{\omega_i^2}\right)\left(\Delta_{c_i}\tilde{G}_1\right)\right| \\ &\quad + \sup_i \left|\frac{1}{T}\frac{1}{\omega_i^2}\left(\Delta_{c_i}\tilde{G}_0\right)'\left(\Delta_{c_i}\tilde{G}_1\right)\right|.\end{aligned}$$

By Lemma 12(a), we have

$$\sup_i \left|\frac{1}{T}\left(\Delta_{c_i}\tilde{G}_0\right)'\left(\Omega^{-1}_{u,i} - \Phi_i\right)\left(\Delta_{c_i}\tilde{G}_1\right)\right| = O\left(\sup_i \frac{\left\|\Delta_{c_i}\tilde{G}_0\right\|}{T}\right) = O\left(\frac{1}{T^{1/2}}\right).$$

By Lemma 11 and Condition 1, we have

$$\sup_i \left|\frac{1}{T}\left(\Delta_{c_i}\tilde{G}_0\right)'\left(\Phi_i - \frac{1}{\omega_i^2}\right)\left(\Delta_{c_i}\tilde{G}_1\right)\right| = O\left(\frac{1}{T^{1/2}}\right).$$

Finally, the last term is

$$\sup_i \left|\frac{1}{T}\frac{1}{\omega_i^2}\left(\Delta_{c_i}\tilde{G}_0\right)'\left(\Delta_{c_i}\tilde{G}_1\right)\right| = \sup_i \left|B^*_{12,iT}(\mathbb{C})\right| = O\left(\frac{1}{T^{1/2}}\right),$$

as required. ■

Part (b) is an immediate corollary of Part (a). ■

Part (c): First notice that under Condition 1 we have

$$\begin{aligned}0 &< M_l \leq \inf_i B_{kk,iT}(\mathbb{C}) = \left\|\frac{1}{T}\left(\Delta_{c_i}\tilde{G}_{k-1}\right)\right\|^2 \frac{1}{\sup_i \lambda_{\max}(\Omega_{u,i})} \\ &\leq B_{kk,iT}(\mathbb{C}) \leq \sup_i B_{kk,iT}(\mathbb{C}) = \left\|\frac{1}{T}\left(\Delta_{c_i}\tilde{G}_{k-1}\right)\right\|^2 \frac{1}{\inf_i \lambda_{\min}(\Omega_{u,i})} \leq M_u < \infty,\end{aligned}$$

where $k=1,2$. It follows immediately that

$$\sup_i \left\|\tilde{B}_{iT}(\mathbb{C})^{-1}\right\| \leq \frac{1}{\inf_i B_{11,iT}(\mathbb{C})} + \frac{1}{\inf_i B_{22,iT}(\mathbb{C})} \leq M,$$

as required. Also, the desired result follows since

$$\begin{aligned}
\sup_i \left\|B_{iT}(\mathbb{C})^{-1}\right\| &= \sup_i \left\| \frac{1}{\det(B_{iT}(\mathbb{C}))} \begin{pmatrix} B_{22,iT}(\mathbb{C}) & -B_{12,iT}(\mathbb{C}) \\ -B_{12,iT}(\mathbb{C}) & B_{11,iT}(\mathbb{C}) \end{pmatrix} \right\| \\
&\leq \frac{\sup_i \|B_{iT}(\mathbb{C})\|}{\inf_i B_{11,iT}(\mathbb{C}) \inf_i B_{22,iT}(\mathbb{C}) - \sup_i B_{12,iT}(\mathbb{C})^2} \\
&= \frac{\sup_i \left\|\tilde{B}_{iT}(\mathbb{C})\right\| + o(1)}{\inf_i B_{11,iT}(\mathbb{C}) \inf_i B_{22,iT}(\mathbb{C}) + o(1)} \leq M,
\end{aligned}$$

where the second equality holds by Part (a). ■

Part (d): The desired result $\sup_i E\|A^*_{iT}(\mathbb{C})\|^2 \leq M$ follows from (24) and by direct calculation. For the second desired result, notice that $E\|A_{iT}(\mathbb{C})\|^2 = E\|A_{1,iT}(\mathbb{C})\|^2 + E\|A_{2,iT}(\mathbb{C})\|^2$ . First, $\sup_i E\|A_{2,iT}(\mathbb{C})\|^2 \leq M$ since $E\|A_{2,iT}(\mathbb{C})\|^2 \leq 2E\left\|A^*_{2,iT}(\mathbb{C})\right\|^2 + 2E\left\|A_{2,iT}(\mathbb{C}) - A^*_{2,iT}(\mathbb{C})\right\|^2 \leq M$ by $\sup_i E\|A^*_{iT}(\mathbb{C})\|^2 \leq M$ and by Part (g) which we prove later. Next, by definition,

$$\begin{aligned}
A_{1,iT}(\mathbb{C}) &= \frac{1}{T^{1/2}} \left(\Delta_{c_i}\tilde{G}_0\right)' \Omega^{-1}_{u,i} \left(\Delta_{c_i}\underline{Y}_i\right) \\
&= \frac{(c_i - \theta_i)}{n^{1/4}} \frac{1}{T^{3/2}} \left(\Delta_{c_i}\tilde{G}_0\right)' \Omega^{-1}_{u,i}\underline{Y}_{-1,i} + \frac{1}{T^{1/2}} \left(\Delta_{c_i}\tilde{G}_0\right)' \Omega^{-1}_{u,i}\underline{U}_i \\
&= I_i + II_i, \text{ say.}
\end{aligned}$$

Since $\underline{Y}_{-1,i} = A_i\underline{U}_i$, where $A_i$ is defined above Lemma 10, we have

$$\begin{aligned}
\sup_i E\left(I_i^2\right) &= \sup_i E\left( \frac{(c_i-\theta_i)^2}{n^{1/2}\omega_i^2} \frac{1}{T^3} \left(\Delta_{c_i}\tilde{G}_0\right)' \Omega^{-1}_{u,i} A_i \Omega_{u,i} A_i \Omega^{-1}_{u,i} \left(\Delta_{c_i}\tilde{G}_0\right) \right) \\
&\leq M \frac{1}{n^{1/2}} \left( \sup_i \left\| \frac{\Delta_{c_i}\tilde{G}_0}{T^{1/2}} \right\|^2 \right) \left( \sup_i \left\| \frac{A_i}{T} \right\|^2 \right) \left( \sup_i \left\|\Omega^{-1}_{u,i}\right\|_o^2 \right) \left( \sup_i \|\Omega_{u,i}\|_o \right) \\
&= O\left(\frac{1}{n^{1/2}}\right) = o(1),
\end{aligned}$$

and

$$\begin{aligned}
\sup_i E\left(II_i^2\right) &= \sup_i \frac{1}{T} \left(\Delta_{c_i}\tilde{G}_0\right)' \Omega^{-1}_{u,i} \left(\Delta_{c_i}\tilde{G}_0\right) \\
&\leq \left( \sup_i \left\| \frac{\Delta_{c_i}\tilde{G}_0}{T^{1/2}} \right\|^2 \right) \left( \sup_i \left\|\Omega^{-1}_{u,i}\right\|_o \right) = O(1).
\end{aligned}$$

Therefore, we have

$$\sup_i E\left\|A_{1,iT}^*\left(\mathbb{C}\right)\right\|^2 \leq M,$$

as required. ■

Part (e): Notice that

$$\begin{aligned}
& B_{11,iT}\left(\mathbb{C}\right)-B_{11,iT}\left(0\right) \\
= \; & \frac{1}{T}\left(\Delta_{c_i}\tilde{G}_0-\Delta\tilde{G}_0\right)'\Omega_{u,i}^{-1}\left(\Delta_{c_i}\tilde{G}_0-\Delta\tilde{G}_0\right)-\frac{2}{T}\left(\Delta_{c_i}\tilde{G}_0-\Delta\tilde{G}_0\right)'\Omega_{u,i}^{-1}\left(\Delta\tilde{G}_0\right).
\end{aligned}$$

The required result follows since

$$\begin{aligned}
& \sup_i\left|B_{11,iT}\left(\mathbb{C}\right)-B_{11,iT}\left(0\right)\right| \\
\leq \; & \frac{1}{T}\left(\sup_i\left\|\Delta_{c_i}\tilde{G}_0-\Delta\tilde{G}_0\right\|^2\right)\left(\sup_i\left\|\Omega_{u,i}^{-1}\right\|_o\right)+\frac{2}{T}\left(\sup_i\left\|\Delta_{c_i}\tilde{G}_0-\Delta\tilde{G}_0\right\|\right)\left(\sup_i\left\|\Omega_{u,i}^{-1}\right\|_o\right)\left\|\Delta\tilde{G}_0\right\| \\
= \; & \frac{1}{T}O\left(\frac{1}{n^{1/2}}\right)O\left(1\right)+\frac{1}{T}O\left(\frac{1}{n^{1/4}}\right)O\left(1\right)O\left(T^{1/2}\right)=O\left(\frac{1}{n^{1/4}T^{1/2}}\right),
\end{aligned}$$

as required. ■

Part (f) follows by Lemma 12b(2). ■

Part (g): By definition, we have

$$\begin{aligned}
& A_{2,iT}\left(\mathbb{C}\right)-A_{2,iT}^*\left(\mathbb{C}\right) \\
= \; & \frac{\left(c_i-\theta_i\right)}{n^{1/4}}\frac{1}{T^{3/2}}\left(\Delta_{c_i}\tilde{G}_1\right)'\left(\Omega_{u,i}^{-1}-\frac{1}{\omega_i^2}\right)\underline{Y}_{-1,i}+\frac{1}{T^{1/2}}\left(\Delta_{c_i}\tilde{G}_1\right)'\left(\Omega_{u,i}^{-1}-\frac{1}{\omega_i^2}\right)\underline{U}_i \\
= \; & \frac{\left(c_i-\theta_i\right)}{n^{1/4}\omega_i}\frac{1}{T^{3/2}}\left(\left(\Delta_{c_i}\tilde{G}_1\right)'R_i\right)\Omega_{u,i}^{-1/2}A_i\underline{U}_i+\frac{1}{\omega_iT^{1/2}}\left(\left(\Delta_{c_i}\tilde{G}_1\right)'R_i\right)\Omega_{u,i}^{-1/2}\underline{U}_i \\
= \; & I_i+II_i, \text{ say,}
\end{aligned}$$

where the second equality holds since $\underline{Y}_{-1,i}=A_i\underline{U}_i$, where $A_i$ is defined above Lemma 10, and $R_i$ is defined in $(20)$.

$$\begin{aligned}
\sup_i E\left(I_i^2\right) &= \sup_i E\left(\frac{\left(c_i-\theta_i\right)^2}{n^{1/2}\omega_i^2}\frac{1}{T^3}\left(\left(\Delta_{c_i}\tilde{G}_1\right)'R_i\right)\Omega_{u,i}^{-1/2}A_i\Omega_{u,i}A_i\Omega_{u,i}^{-1/2}\left(R_i'\left(\Delta_{c_i}\tilde{G}_1\right)\right)\right) \\
&\leq M\frac{1}{n^{1/2}}\left(\sup_i\frac{1}{T}\left\|\left(\Delta_{c_i}\tilde{G}_1\right)'R_i\right\|^2\right)\left(\sup_i\left\|\frac{A_i}{T}\right\|^2\right)\left(\sup_i\left\|\Omega_{u,i}^{-1}\right\|_o\right)\left(\sup_i\left\|\Omega_{u,i}\right\|_o\right) \\
&= O\left(\frac{1}{n^{1/2}T^{1/2}}\right),
\end{aligned}$$

and

$$\sup_i E\left(II_i^2\right)=\sup_i\frac{1}{\omega_i^2}\sup_i\frac{1}{T}\left\|\left(\Delta_{c_i}\tilde{G}_1\right)'R_i\right\|^2=O\left(\frac{1}{T^{1/2}}\right).$$

Combining the bounds of $\sup_i E\left(I_i^2\right)$ and $\sup_i E\left(II_i^2\right)$, we have the desired result for Part (g). ■

## 5.5 Proof of (18)

The required result follows if we show

$$\begin{aligned}\sum_{i=1}^{n}\left[A_{1,iT}\left(\mathbb{C}\right)^{2}B_{11,iT}\left(\mathbb{C}\right)^{-1}-A_{1,iT}\left(0\right)^{2}B_{11,iT}\left(0\right)^{-1}\right] &= o_{p}\left(1\right), \\ \sum_{i=1}^{n}\left[A_{1,iT}^{*}\left(\mathbb{C}\right)^{2}B_{11,iT}^{*}\left(\mathbb{C}\right)^{-1}-A_{1,iT}^{*}\left(0\right)^{2}B_{11,iT}^{*}\left(0\right)^{-1}\right] &= o_{p}\left(1\right).\end{aligned}$$

Notice that

$$\begin{aligned}&\left|\sum_{i=1}^{n}\left[A_{1,iT}\left(\mathbb{C}\right)^{2}B_{11,iT}\left(\mathbb{C}\right)^{-1}-A_{1,iT}\left(0\right)^{2}B_{11,iT}\left(0\right)^{-1}\right]\right| \\ \leq\ & \sum_{i=1}^{n}\left|A_{1,iT}\left(\mathbb{C}\right)^{2}\left(B_{11,iT}\left(\mathbb{C}\right)^{-1}-B_{11,iT}\left(0\right)^{-1}\right)\right|+\sum_{i=1}^{n}\left|A_{1,iT}\left(\mathbb{C}\right)^{2}-A_{1,iT}\left(0\right)^{2}\right|B_{11,iT}\left(0\right)^{-1}.\end{aligned}$$

The first term is bounded by

$$\begin{aligned}& n\left(\sup_{i}B_{11,iT}\left(\mathbb{C}\right)^{-1}\right)\left(\sup_{i}B_{11,iT}\left(0\right)^{-1}\right)\sup_{i}\left|B_{11,iT}\left(\mathbb{C}\right)-B_{11,iT}\left(0\right)\right|\left(\frac{1}{n}\sum_{i=1}^{n}A_{1,iT}\left(\mathbb{C}\right)^{2}\right) \\ =\ & nO\left(1\right)O\left(1\right)O\left(\frac{1}{n^{1/4}T^{1/2}}\right)O_{p}\left(1\right)=O_{p}\left(\frac{n^{3/4}}{T^{1/2}}\right)=o_{p}\left(1\right),\end{aligned}$$

where the first equality holds by Lemma 13(c),(d), and (e) and the last equality holds since $\frac{n}{T^{1/2}}=o\left(1\right)$. The second term is bounded by

$$\begin{aligned}& n\left(\frac{1}{n}\sum_{i=1}^{n}\left(A_{1,iT}\left(\mathbb{C}\right)-A_{1,iT}\left(0\right)\right)^{2}\right)^{1/2}\left(\frac{1}{n}\sum_{i=1}^{n}\left(A_{1,iT}\left(\mathbb{C}\right)+A_{1,iT}\left(0\right)\right)^{2}\right)^{1/2}\sup_{i}B_{11,iT}\left(0\right)^{-1} \\ =\ & nO_{p}\left(\frac{1}{n^{1/8}T^{1/2}}\right)O_{p}\left(1\right)O\left(1\right)=O_{p}\left(\frac{n^{7/8}}{T^{1/2}}\right)=o_{p}\left(1\right),\end{aligned}$$

where the first equality holds by Lemma 13(c),(d), and $\sup_{i}E\left(A_{1,iT}\left(\mathbb{C}\right)-A_{1,iT}\left(0\right)\right)^{2}=O\left(\frac{1}{n^{1/2}T}\right)$, and the last equality holds since $\frac{n}{T^{1/2}}=o\left(1\right)$. Combining these two, we have the required result

$$\sum_{i=1}^{n}\left[A_{1,iT}\left(\mathbb{C}\right)^{2}B_{11,iT}\left(\mathbb{C}\right)^{-1}-A_{1,iT}\left(0\right)^{2}B_{11,iT}\left(0\right)^{-1}\right]=o_{p}\left(1\right).$$

The second required result for Step 2 follows in similar fashion and we omit it.
■

## 5.6 Proof of (19)

The required result follows, if we show

$$\begin{aligned} & \sum_{i=1}^{n}\left[A_{iT}\left(\mathbb{C}\right)' B_{iT}\left(\mathbb{C}\right)^{-1} A_{iT}\left(\mathbb{C}\right) - A_{iT}\left(0\right) B_{iT}\left(0\right)^{-1} A_{iT}\left(0\right)\right] \\ & -\sum_{i=1}^{n}\left[A_{iT}^{*}\left(\mathbb{C}\right)' B_{iT}^{*}\left(\mathbb{C}\right)^{-1} A_{iT}^{*}\left(\mathbb{C}\right) - A_{iT}^{*}\left(0\right)' B_{iT}^{*}\left(0\right)^{-1} A_{iT}^{*}\left(0\right)\right] \\ = \; & o_p\left(1\right), \end{aligned}$$

which will be established by the following three steps.

- Step 1: We show

$$\begin{aligned} & \sum_{i=1}^{n}\left[A_{iT}\left(\mathbb{C}\right)' B_{iT}\left(\mathbb{C}\right)^{-1} A_{iT}\left(\mathbb{C}\right) - A_{iT}\left(0\right) B_{iT}\left(0\right)^{-1} A_{iT}\left(0\right)\right] \\ & -\sum_{i=1}^{n}\left[A_{iT}^{*}\left(\mathbb{C}\right)' B_{iT}^{*}\left(\mathbb{C}\right)^{-1} A_{iT}^{*}\left(\mathbb{C}\right) - A_{iT}^{*}\left(0\right)' B_{iT}^{*}\left(0\right)^{-1} A_{iT}^{*}\left(0\right)\right] \\ = \; & \sum_{i=1}^{n}\left[A_{iT}\left(\mathbb{C}\right)' \tilde{B}_{iT}\left(\mathbb{C}\right)^{-1} A_{iT}\left(\mathbb{C}\right) - A_{iT}\left(0\right) \tilde{B}_{iT}\left(0\right)^{-1} A_{iT}\left(0\right)\right] \\ & -\sum_{i=1}^{n}\left[A_{iT}^{*}\left(\mathbb{C}\right)' \tilde{B}_{iT}^{*}\left(\mathbb{C}\right)^{-1} A_{iT}^{*}\left(\mathbb{C}\right) - A_{iT}^{*}\left(0\right)' \tilde{B}_{iT}^{*}\left(0\right)^{-1} A_{iT}^{*}\left(0\right)\right] + o_p\left(1\right). \end{aligned}$$

- Step 2: By (18) we have

$$\begin{aligned} = \; & \sum_{i=1}^{n}\left[A_{iT}\left(\mathbb{C}\right)' \tilde{B}_{iT}\left(\mathbb{C}\right)^{-1} A_{iT}\left(\mathbb{C}\right) - A_{iT}\left(0\right) \tilde{B}_{iT}\left(0\right)^{-1} A_{iT}\left(0\right)\right] \\ & -\sum_{i=1}^{n}\left[A_{iT}^{*}\left(\mathbb{C}\right)' \tilde{B}_{iT}^{*}\left(\mathbb{C}\right)^{-1} A_{iT}^{*}\left(\mathbb{C}\right) - A_{iT}^{*}\left(0\right)' \tilde{B}_{iT}^{*}\left(0\right)^{-1} A_{iT}^{*}\left(0\right)\right] \\ = \; & \sum_{i=1}^{n}\left[A_{2,iT}\left(\mathbb{C}\right)^{2} B_{22,iT}\left(\mathbb{C}\right)^{-1} - A_{2,iT}^{*}\left(\mathbb{C}\right)^{2} B_{22,iT}^{*}\left(\mathbb{C}\right)^{-1}\right] \\ & -\sum_{i=1}^{n}\left[A_{2,iT}\left(0\right)^{2} B_{22,iT}\left(0\right)^{-1} - A_{2,iT}^{*}\left(0\right)^{2} B_{22,iT}^{*}\left(0\right)^{-1}\right] + o_p\left(1\right). \end{aligned}$$

- Step 3: We show

$$\begin{aligned} \sum_{i=1}^{n}\left[A_{2,iT}\left(\mathbb{C}\right)^{2} B_{22,iT}\left(\mathbb{C}\right)^{-1} - A_{2,iT}^{*}\left(\mathbb{C}\right)^{2} B_{22,iT}^{*}\left(\mathbb{C}\right)^{-1}\right] &= o_p\left(1\right) \\ \sum_{i=1}^{n}\left[A_{2,iT}\left(0\right)^{2} B_{22,iT}\left(0\right)^{-1} - A_{2,iT}^{*}\left(0\right)^{2} B_{22,iT}^{*}\left(0\right)^{-1}\right] &= o_p\left(1\right). \end{aligned}$$

Proof of Step 1: Notice that since $B_{iT}^{*}(\mathbb{C})$ is a diagonal matrix,

$$\sum_{i=1}^{n} A_{iT}^{*}(\mathbb{C})'\left(B_{iT}^{*}(\mathbb{C})^{-1}-\tilde{B}_{iT}^{*}(\mathbb{C})^{-1}\right)A_{iT}^{*}(\mathbb{C})=0.$$

Then, the required result for Step 1 follows if we show

$$\sum_{i=1}^{n} A_{iT}(\mathbb{C})'\left(B_{iT}(\mathbb{C})^{-1}-\tilde{B}_{iT}(\mathbb{C})^{-1}\right)A_{iT}(\mathbb{C})=o_{p}(1).$$

The required result follows since

$$\begin{aligned}
& \left|\sum_{i=1}^{n} A_{iT}(\mathbb{C})'\left(B_{iT}(\mathbb{C})^{-1}-\tilde{B}_{iT}(\mathbb{C})^{-1}\right)A_{iT}(\mathbb{C})\right| \\
= & \left|\sum_{i=1}^{n} A_{iT}(\mathbb{C})'\left(B_{iT}(\mathbb{C})^{-1}\left(\tilde{B}_{iT}(\mathbb{C})-B_{iT}(\mathbb{C})\right)\tilde{B}_{iT}(\mathbb{C})^{-1}\right)A_{iT}(\mathbb{C})\right| \\
\leq & \sum_{i=1}^{n}\left\|A_{iT}(\mathbb{C})\right\|^{2}\left\|B_{iT}(\mathbb{C})^{-1}\right\|\left\|\tilde{B}_{iT}(\mathbb{C})^{-1}\right\|\left\|\tilde{B}_{iT}(\mathbb{C})-B_{iT}(\mathbb{C})\right\| \\
\leq & n\left(\sup_{i}\left\|B_{iT}(\mathbb{C})^{-1}\right\|\right)\left(\sup_{i}\left\|\tilde{B}_{iT}(\mathbb{C})^{-1}\right\|\right)\left(\sup_{i}\left\|\tilde{B}_{iT}(\mathbb{C})-B_{iT}(\mathbb{C})\right\|\right)\left(\frac{1}{n}\sum_{i=1}^{n}\left\|A_{iT}(\mathbb{C})\right\|^{2}\right) \\
= & nO(1)\,O(1)\,O\left(\frac{1}{T^{1/2}}\right)O_{p}(1)=O_{p}\left(\frac{n}{T^{1/2}}\right)=o_{p}(1),
\end{aligned}$$

where the last line holds by Lemma 13(b),(c), and (d) and the condition $\frac{n}{T^{1/4}} \to 0$. ■

Proof of Step 3: We show

$$\sum_{i=1}^{n}\left[A_{2,iT}(\mathbb{C})^{2}B_{22,iT}(\mathbb{C})^{-1}-A_{2,iT}^{*}(\mathbb{C})^{2}B_{22,iT}^{*}(\mathbb{C})^{-1}\right]=o_{p}(1).$$

The other required result $\sum_{i=1}^{n}\left[A_{2,iT}(0)^{2}B_{22,iT}(0)^{-1}-A_{2,iT}^{*}(0)^{2}B_{22,iT}^{*}(0)^{-1}\right]= o_{p}(1)$ follows in similar fashion and we omit the derivation. Notice that

$$\begin{aligned}
& \left|\sum_{i=1}^{n}\left[A_{2,iT}(\mathbb{C})^{2}B_{22,iT}(\mathbb{C})^{-1}-A_{2,iT}^{*}(\mathbb{C})^{2}B_{22,iT}^{*}(\mathbb{C})^{-1}\right]\right| \\
\leq & \left|\sum_{i=1}^{n}\left(A_{2,iT}(\mathbb{C})^{2}-A_{2,iT}^{*}(\mathbb{C})^{2}\right)B_{22,iT}(\mathbb{C})^{-1}\right|+\left|\sum_{i=1}^{n}A_{2,iT}^{*}(\mathbb{C})^{2}\left(B_{22,iT}(\mathbb{C})^{-1}-B_{22,iT}^{*}(\mathbb{C})^{-1}\right)\right|
\end{aligned}$$

For the first term, we have

$$
\begin{aligned}
& \left| \sum_{i=1}^{n} \left( A_{2,iT}(\mathbb{C})^2 - A_{2,iT}^{*}(\mathbb{C})^2 \right) B_{22,iT}(\mathbb{C})^{-1} \right| \\
\leq \;& n \left( \frac{1}{n} \sum_{i=1}^{n} \left( A_{2,iT}(\mathbb{C}) - A_{2,iT}^{*}(\mathbb{C}) \right)^2 \right)^{1/2} \left( \frac{1}{n} \sum_{i=1}^{n} \left( A_{2,iT}(\mathbb{C}) + A_{2,iT}^{*}(\mathbb{C}) \right)^2 \right)^{1/2} \sup_i B_{22,iT}(\mathbb{C})^{-1} \\
= \;& n O_p\left( \frac{1}{T^{1/4}} \right) O_p(1)\, O(1) = O_p\left( \frac{n}{T^{1/4}} \right) = o_p(1),
\end{aligned}
$$

where the first equality holds by Lemma 13(c),(d), and (g) and the last equality holds by the condition $\frac{n}{T^{1/4}} \to 0$. For the second term, notice that

$$
\begin{aligned}
& \left| \sum_{i=1}^{n} \left[ A_{2,iT}^{*}(\mathbb{C})^2 \left( B_{22,iT}(\mathbb{C})^{-1} - B_{22,iT}^{*}(\mathbb{C})^{-1} \right) \right] \right| \\
= \;& \left| \sum_{i=1}^{n} \left[ A_{2,iT}^{*}(\mathbb{C})^2 B_{22,iT}(\mathbb{C})^{-1} \left( B_{22,iT}(\mathbb{C}) - B_{22,iT}^{*}(\mathbb{C}) \right) B_{22,iT}^{*}(\mathbb{C})^{-1} \right] \right| \\
\leq \;& n \left( \frac{1}{n} \sum_{i=1}^{n} A_{2,iT}^{*}(\mathbb{C})^2 \right) \left( \sup_i B_{22,iT}(\mathbb{C})^{-1} \right) \left( \sup_i B_{22,iT}^{*}(\mathbb{C})^{-1} \right) \sup_i \left| B_{22,iT}(\mathbb{C}) - B_{22,iT}^{*}(\mathbb{C}) \right| \\
= \;& n O_p(1)\, O(1)\, O(1)\, O\left( \frac{1}{T^{1/2}} \right) = O_p\left( \frac{n}{T^{1/2}} \right) = o_p(1),
\end{aligned}
$$

where the second equality holds by Lemma 13(c),(d), and (f) and the last equality holds by the condition $\frac{n}{T^{1/4}} \to 0$. Then, we have all the desired results for Part (c). ■

# 6 Proofs of the Limit Distribution Results: Theorems 3, 5, and 7

In this section we provide proofs of Theorems 3, 5, and 7. These proofs are very similar to the proofs of the corresponding results in MPP and we therefore provide just an outline of the proofs here.

## 6.1 Proof of Theorem 3

Since $\Delta y_{it} = -\frac{\theta_i}{n^{1/2}T} y_{it-1} + u_{it}$, we can write

$$
\begin{aligned}
& V_{nT}(\mathbb{C}) \\
= & \sum_{i=1}^{n} \frac{1}{\omega_i^2} \left[ y_{i1}^2 + \sum_{t=2}^{T} (\Delta_{c_i} y_{it})^2 \right] - \sum_{i=1}^{n} \frac{1}{\omega_i^2} \left[ y_{i1}^2 + \sum_{t=2}^{T} (\Delta y_{it})^2 \right] - \frac{2}{n^{1/2}} \sum_{i=1}^{n} c_i \frac{\lambda_i}{\omega_i^2} - \frac{1}{2}\mu_{c,2} \\
= & \frac{2}{n^{1/2}T} \sum_{i=1}^{n} \frac{c_i}{\omega_i^2} \sum_{t=2}^{T} \Delta y_{it} y_{it-1} + \frac{1}{nT^2} \sum_{i=1}^{n} \frac{c_i^2}{\omega_i^2} \sum_{t=2}^{T} y_{it-1}^2 - \frac{2}{n^{1/2}} \sum_{i=1}^{n} c_i \frac{\lambda_i}{\omega_i^2} - \frac{1}{2}\mu_{c,2} \\
= & -\frac{2}{nT^2} \sum_{i=1}^{n} \frac{c_i \theta_i}{\omega_i^2} \sum_{t=2}^{T} y_{it-1}^2 + \frac{2}{n^{1/2}T} \sum_{i=1}^{n} \frac{c_i}{\omega_i^2} \sum_{t=2}^{T} u_{it} y_{it-1} - \frac{2}{n^{1/2}} \sum_{i=1}^{n} c_i \frac{\lambda_i}{\omega_i^2} \\
& + \frac{1}{nT^2} \sum_{i=1}^{n} \frac{c_i^2}{\omega_i^2} \sum_{t=2}^{T} y_{it-1}^2 - \frac{1}{2}\mu_{c,2}.
\end{aligned}
$$

Direct calculation shows that under the assumptions of the theorem, we have the following joint limits

$$
\begin{aligned}
-\frac{2}{nT^2} \sum_{i=1}^{n} \sum_{t=1}^{T} \frac{c_i \theta_i}{\omega_i^2} y_{it-1}^2 &\rightarrow_p -E(c_i \theta_i), \\
\frac{1}{nT^2} \sum_{i=1}^{n} \frac{c_i^2}{\omega_i^2} \sum_{t=1}^{T} y_{it-1}^2 &\rightarrow_p \frac{1}{2}\mu_{c,2},
\end{aligned}
$$

and CLT

$$
\frac{2}{n^{1/2}} \sum_{i=1}^{n} \frac{c_i}{\omega_i^2} \left( \frac{1}{T} \sum_{t=1}^{T} u_{it} y_{it-1} - \lambda_i \right) \Rightarrow N\left(0, 2\mu_{c,2}\right),
$$

thereby giving the required result. ■

## 6.2 Proof Theorem 5

For the required result of the theorem, it is enough to show that

$$
V_{fe1,nT}(\mathbb{C}) = V_{nT}(\mathbb{C}) + o_p(1).
$$

Let $\hat{b}_{0i}(c_i) = \left(\Delta_{c_i} G_0' \Delta_{c_i} G_0\right)^{-1} \left(\Delta_{c_i} G_0' \Delta_{c_i} \underline{Z}_i\right)$. Then $\underline{Z}_i - G_0 \hat{b}_{0i}(c_i) = \underline{Y}_i - G_0 \left(\hat{b}_{0i}(c_i) - b_{0i}\right)$, and we can rewrite $V_{fe1,nT}(\mathbb{C})$ as

$$
\begin{aligned}
& V_{fe1,nT}(\mathbb{C}) \\
= & \sum_{i=1}^{n} \frac{1}{\omega_i^2} \left[ \begin{array}{c} \left(\Delta_{c_i} \underline{Y}_i - \Delta_{c_i} G_0 \left(\hat{b}_{0i}(c_i) - b_{0i}\right)\right)' \left(\Delta_{c_i} \underline{Y}_i - \Delta_{c_i} G_0 \left(\hat{b}_{0i}(c_i) - b_{0i}\right)\right) \\ - \left(\Delta \underline{Y}_i - \Delta G_0 \left(\hat{b}_{0i}(c_i) - b_{0i}\right)\right)' \left(\Delta \underline{Y}_i - \Delta G_0 \left(\hat{b}_{0i}(c_i) - b_{0i}\right)\right) \end{array} \right] \\
& - \frac{2}{n^{1/2}} \sum_{i=1}^{n} c_i \frac{\lambda_i}{\omega_i^2} - \frac{1}{2}\mu_{c,2} \\
= & V_{nT}(\mathbb{C}) + V_{fe11,nT}(\mathbb{C}),
\end{aligned}
$$

where

$$V_{fe11,nT}(\mathbb{C}) = \sum_{i=1}^{n} \frac{1}{\omega_i^2} \left[ \begin{array}{c} \left(\Delta \underline{Y}_i' \Delta G_0\right) \left(\Delta G_0' \Delta G_0\right)^{-1} \left(\Delta G_0' \Delta \underline{Y}_i\right) \\ -\left(\Delta_{c_i} \underline{Y}_i' \Delta_{c_i} G_0\right) \left(\Delta_{c_i} G_0' \Delta_{c_i} G_0\right)^{-1} \left(\Delta_{c_i} G_0' \Delta_{c_i} \underline{Y}_i\right) \end{array} \right].$$

We can follow the proof on pages 449-450 of MPP and deduce that

$$V_{fe11,nT}(\mathbb{C}) = o_p(1)$$

as $n, T \to \infty$ with $\frac{n}{T} \to 0$, which proves the desired result. ■

## 6.3 Proof of Theorem 7

The required result for Theorem 7 is a consequence of the following two lemmas. ■

**Lemma 14** *Assume Condition 1. Then, as $n, T \to \infty$ with $\frac{n}{T} \to 0$, we have*

$$\begin{aligned} & V_{fe2,nT}(\mathbb{C}) \\ = \; & \frac{1}{n^{1/4}} \sum_{i=1}^{n} \frac{c_i}{\omega_i^2} \left[ \frac{2}{T} \sum_{t=2}^{T} \Delta y_{it} y_{it-1} - \left(\frac{y_{iT}}{\sqrt{T}}\right)^2 + \left(\frac{y_{i1}}{\sqrt{T}}\right)^2 + \sigma_i^2 \right] \\ & + \frac{1}{n^{1/2}} \sum_{i=1}^{n} \frac{c_i^2}{\omega_i^2} \left[ \begin{array}{c} \frac{1}{T^2} \sum_{t=2}^{T} y_{it-1}^2 - 2\left(\frac{y_{iT}}{\sqrt{T}}\right)\left(\frac{1}{T\sqrt{T}} \sum_{t=2}^{T} \frac{t}{T} y_{it-1}\right) \\ + \frac{1}{3}\left(\frac{y_{iT}}{\sqrt{T}}\right)^2 + \omega_i^2 \omega_{p2T} \end{array} \right] \\ & + \frac{1}{n} \sum_{i=1}^{n} \frac{c_i^4}{\omega_i^2} \left[ \begin{array}{c} -\left(\frac{1}{T\sqrt{T}} \sum_{t=2}^{T} \frac{t}{T} y_{it-1}\right)^2 + \frac{2}{3}\left(\frac{y_{iT}}{\sqrt{T}}\right)\left(\frac{1}{T\sqrt{T}} \sum_{t=2}^{T} \frac{t}{T} y_{it-1}\right) \\ -\frac{1}{9}\left(\frac{y_{iT}}{\sqrt{T}}\right)^2 + \omega_i^2 \omega_{p4T} \end{array} \right] \\ & + o_p(1). \end{aligned}$$

**Proof.** The proof is similar to the proof of Lemma 11 of MPP and is omitted. ■

**Lemma 15** *Assume Condition 1. Then, as $n, T \to \infty$ with $\frac{n}{T} \to 0$, the following hold:*

(a) $\frac{1}{n^{1/4}} \sum_{i=1}^{n} \frac{c_i}{\omega_i^2} \left[ \frac{2}{T} \sum_{t=1}^{T} \Delta y_{it} y_{it-1} - \left(\frac{y_{iT}}{\sqrt{T}}\right)^2 + \left(\frac{y_{i1}}{\sqrt{T}}\right)^2 + \sigma_i^2 \right] = o_p(1);$

(b) $\frac{1}{n^{1/2}} \sum_{i=1}^{n} \frac{c_i^2}{\omega_i^2} \left[ \begin{array}{c} \left(\frac{1}{T^2} \sum_{t=2}^{T} y_{it-1}^2 - \omega_i^2 \frac{1}{T} \sum_{t=2}^{T} \frac{t-1}{T}\right) + \frac{1}{3}\left\{\left(\frac{y_{iT}}{\sqrt{T}}\right)^2 - \omega_i^2\right\} \\ -\left\{2\left(\frac{y_{iT}}{\sqrt{T}}\right)\left(\frac{1}{T\sqrt{T}} \sum_{t=2}^{T} \frac{t}{T} y_{it-1}\right) - \omega_i^2 \frac{2}{T} \sum_{t=2}^{T} \left(\frac{t}{T}\right)\left(\frac{t-1}{T}\right)\right\} \end{array} \right] \Rightarrow$
$N\left(-\frac{1}{90} E\left(c_i^2 \theta_i^2\right), \frac{1}{45} E\left(c_i^4\right)\right);$

(c) $\frac{1}{n}\sum_{i=1}^{n}\frac{c_i^4}{\omega_i^2}\left[\begin{array}{c} -\left(\frac{1}{T\sqrt{T}}\sum_{t=2}^{T}\frac{t}{T}y_{it-1}\right)^2+\frac{2}{3}\left(\frac{y_{iT}}{\sqrt{T}}\right)\left(\frac{1}{T\sqrt{T}}\sum_{t=2}^{T}\frac{t}{T}y_{it-1}\right) \\ -\frac{1}{9}\left(\frac{y_{iT}}{\sqrt{T}}\right)^2+\omega_i^2\omega_{p4T} \end{array}\right]=$ $o_p(1)$.

**Proof.** The proofs of Parts (b) and (c) are similar to those of Lemma 12 (b) and (c) and are skipped.

Part (a): First, notice from

$$y_{it}^2-y_{it-1}^2=\left(\rho_i^2-1\right)y_{it-1}^2+2\rho_i y_{it-1}u_{it}+u_{it}^2 \text{ for } t\geq 2,$$

that

$$\left(\frac{y_{iT}}{\sqrt{T}}\right)^2-\left(\frac{y_{i1}}{\sqrt{T}}\right)^2=\left(\rho_i^2-1\right)\frac{1}{T}\sum_{t=2}^{T}y_{it-1}^2+2\rho_i\frac{1}{T}\sum_{t=2}^{T}y_{it-1}u_{it}+\frac{1}{T}\sum_{t=2}^{T}u_{it}^2.$$

Since $\Delta y_{it}=(\rho_i-1)y_{it-1}+u_{it}$, we have

$$2\frac{1}{T}\sum_{t=2}^{T}\Delta y_{it}y_{it-1}=2(\rho_i-1)\frac{1}{T}\sum_{t=2}^{T}y_{it-1}^2+2\frac{1}{T}\sum_{t=2}^{T}y_{it-1}u_{it}.$$

Then,

$$\begin{aligned} & \frac{1}{n^{1/4}}\sum_{i=1}^{n}\frac{c_i}{\omega_i^2}\left[\frac{2}{T}\sum_{t=2}^{T}\Delta y_{it}y_{it-1}-\left(\frac{y_{iT}}{\sqrt{T}}\right)^2+\left(\frac{y_{i1}}{\sqrt{T}}\right)^2+\sigma_i^2\right] \\ = & \frac{1}{n^{1/4}}\sum_{i=1}^{n}\frac{c_i}{\omega_i^2}\left[-(\rho_i-1)^2\frac{1}{T}\sum_{t=2}^{T}y_{it-1}^2+2(1-\rho_i)\frac{1}{T}\sum_{t=1}^{T}y_{it-1}u_{it}-\left(\frac{1}{T}\sum_{t=1}^{T}u_{it}^2-\sigma_i^2\right)\right]. \end{aligned}$$

Under the assumptions of the lemma,

$$\frac{1}{n^{1/4}}\sum_{i=1}^{n}\frac{c_i}{\omega_i^2}(\rho_i-1)^2\frac{1}{T}\sum_{t=2}^{T}y_{it-1}^2=\frac{n^{1/4}}{T}\left(\frac{1}{n}\sum_{i=1}^{n}c_i\theta_i^2\left(\frac{1}{T^2\omega_i^2}\sum_{t=2}^{T}y_{it-1}^2\right)\right)=O_p\left(\frac{n^{1/4}}{T}\right),$$

$$\frac{1}{n^{1/4}}\sum_{i=1}^{n}\frac{c_i}{\omega_i^2}(1-\rho_i)\frac{1}{T}\sum_{t=2}^{T}y_{it-1}u_{it}=\frac{1}{T}\frac{2}{n^{1/2}}\sum_{i=1}^{n}\theta_i\left(\frac{1}{T\omega_i^2}\sum_{t=2}^{T}y_{it-1}u_{it}\right)=O_p\left(\frac{n^{1/2}}{T}\right),$$

and

$$\frac{1}{n^{1/4}}\sum_{i=1}^{n}\frac{c_i}{\omega_i^2}\left(\frac{1}{T}\sum_{t=2}^{T}u_{it}^2-\sigma_i^2\right)=O_p\left(\frac{n^{1/4}}{T^{1/2}}\right),$$

leading to the required result for Part (a). ■

## 7 Proof of Theorem 9

We provide a sketch of the proof. Notice that under Condition 8, the following hold:

$$\begin{aligned}\sup_i \left|\hat{\omega}_i^2 - \omega_i^2\right|, \ \sup_i \left|\hat{\lambda}_i^2 - \lambda_i^2\right|, \ \sup_i \left|\hat{\sigma}_i^2 - \sigma_i^2\right| &= o_p(1) \\ \sum_{i=1}^n \left(\hat{\omega}_i^2 - \omega_i^2\right)^2, \ \sum_{i=1}^n \left(\hat{\lambda}_i^2 - \lambda_i^2\right)^2, \ \sum_{i=1}^n \left(\hat{\sigma}_i^2 - \sigma_i^2\right)^2 &= o_p(1).\end{aligned}$$

Define $\hat{\imath} = \arg\min_{i\in\{1,\ldots,n\}} \hat{\omega}_i^2$ and $i^* = \arg\min_{i\in\{1,\ldots,n\}} \omega_i^2$ Then,

$$\inf_i \hat{\omega}_i^2 - \inf_i \omega_i^2 \geq \hat{\omega}_{\hat{\imath}}^2 - \omega_{\hat{\imath}}^2 \geq -\sup_i \left|\hat{\omega}_i^2 - \omega_i^2\right| = o_p(1)$$

and

$$\inf_i \hat{\omega}_i^2 - \inf_i \omega_i^2 \leq \hat{\omega}_{i^*}^2 - \omega_{i^*}^2 \leq \sup_i \left|\hat{\omega}_i^2 - \omega_i^2\right| = o_p(1).$$

Since $\inf_i \omega_i^2 > 0$ under Condition 1, we have

$$\inf_i \hat{\omega}_i^2 = \inf_i \omega_i^2 + o_p(1) > 0$$

with probability approaching one. These imply that $\hat{\omega}_i^2$ satisfies the properties in Lemmas 8, 10, and 14 of MPP, while $\hat{\lambda}_i^2$ and $\hat{\sigma}_i^2$ satisfy the properties in Lemmas 8(a),(b), 10(a), and 14(a)-(d) of MPP. The desired results follow by similar arguments to those used in Theorems 8, 10, and 15 of MPP. ■

Table 1. Size of tests robust to serial correlation (nominal size is 5%)

| | N= 10 | | N = 25 | | N = 100 | |
|---|---|---|---|---|---|---|
| | T = 100 | T = 250 | T = 100 | T = 250 | T = 100 | T = 250 |
| | | | Incidental intercepts | | | |
| white noise | 2.8 | 2.9 | 4.2 | 4.2 | 5.3 | 4.9 |
| | 2.8 | 2.8 | 4.1 | 4.0 | 4.5 | 4.5 |
| AR(1) errors | | | | | | |
| -0.2 | 2.9 | 2.6 | 4.2 | 3.9 | 5.1 | 5.0 |
| | 2.7 | 2.7 | 3.8 | 3.7 | 3.9 | 4.5 |
| 0.2 | 2.9 | 2.8 | 4.1 | 4.0 | 6.2 | 5.4 |
| | 2.8 | 2.8 | 4.1 | 3.9 | 5.6 | 5.1 |
| MA(1) errors | | | | | | |
| -0.2 | 4.2 | 4.0 | 7.5 | 7.2 | 14.9 | 15.7 |
| | 2.9 | 2.7 | 3.7 | 3.8 | 4.0 | 4.5 |
| 0.2 | 3.5 | 3.8 | 6.4 | 6.6 | 13.6 | 11.5 |
| | 2.7 | 3.0 | 3.9 | 4.4 | 4.9 | 4.6 |
| | | | Incidental trends | | | |
| white noise | 1.0 | 1.1 | 2.9 | 2.3 | 6.3 | 4.4 |
| | 1.3 | 1.2 | 3.4 | 2.6 | 7.4 | 4.7 |
| AR(1) errors | | | | | | |
| -0.2 | 0.9 | 1.0 | 2.2 | 2.2 | 4.1 | 4.0 |
| | 1.1 | 1.1 | 2.4 | 2.3 | 3.9 | 4.0 |
| 0.2 | 1.0 | 1.0 | 3.5 | 2.7 | 8.8 | 5.1 |
| | 1.5 | 1.2 | 4.7 | 3.1 | 11.6 | 5.7 |
| MA(1) errors | | | | | | |
| -0.2 | 2.2 | 2.0 | 6.7 | 7.8 | 23.0 | 23.4 |
| | 0.8 | 0.9 | 2.3 | 2.4 | 3.1 | 3.3 |
| 0.2 | 2.7 | 2.0 | 7.5 | 6.3 | 27.2 | 19.4 |
| | 2.0 | 1.2 | 4.3 | 3.0 | 9.7 | 6.0 |

Top entry in each cell is with estimated long-run variance and second with known long-run variance