Omitted Proofs for "Incidental Trends and the Power of Panel Unit Root Tests"

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Lemma 5 *Suppose that Assumption 1 is satisfied. Then, as $n, T \to \infty$ with $\frac{n}{T} \to 0$, the following hold.*

(a) $\sum_{i=1}^{n} \left(\tilde{\sigma}_{iT}^2 - \sigma_i^2\right)^2 = o_p(1)$.

(b) $\sup_{1 \le i \le n} \left|\tilde{\sigma}_{iT}^2 - \sigma_i^2\right| = o_p(1)$.

(c) *With probability approaching one, there exists a constant $M > 0$ such that $\inf_i \tilde{\sigma}_{iT}^2 \ge M$.*

Proof

Let $X_{iT} = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left(u_{it}^2 - \sigma_i^2\right)$. Then, $EX_{iT}^2 = Var\left(u_{it}^2\right) \le E\left(u_{it}^4\right) \le \overline{M}$.

Part (a): The required result follows by the Markov inequality and

$$E\left[\sum_{i=1}^{n} \left(\tilde{\sigma}_{iT}^2 - \sigma_i^2\right)^2\right] = \frac{n}{T}\frac{1}{n}\sum_{i=1}^{n} EX_{iT}^2 \le \overline{M}\frac{n}{T} \to 0. \blacksquare$$

Part (b): The required result follows if we show that $\sup_{1 \le i \le n} \left(\tilde{\sigma}_{iT}^2 - \sigma_i^2\right)^2 = o_p(1)$, which holds because $\sup_{1 \le i \le n} \left(\tilde{\sigma}_{iT}^2 - \sigma_i^2\right)^2 \le \sum_{i=1}^{n} \left(\tilde{\sigma}_{iT}^2 - \sigma_i^2\right)^2 = o_p(1)$ by Part (a). $\blacksquare$

Part (c): The required result follows since

$$\inf_{1 \le i \le n} \tilde{\sigma}_{iT}^2 \ge \inf_{1 \le i \le n} \sigma_i^2 - \sup_{1 \le i \le n} \left|\tilde{\sigma}_{iT}^2 - \sigma_i^2\right|$$

and by Part (b) and Assumption 1. $\blacksquare$

Suppose that c_i is a sequence of *iid* random variables with the same support as θ_i and that c_i is independent of u_{it} for all i and t.

Lemma 6 *Suppose that Assumptions 1–3, 6, and 7 hold. Then, the following hold as $(n, T \to \infty)$ with $\frac{n}{T} \to 0$.*

(a) $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} c_i^2 \left[\frac{1}{T^2\sigma_i^2} \sum_{t=1}^{T} \left\{(y_{it} - y_{i0}) - \frac{t}{T}(y_{iT} - y_{i0})\right\}^2 - \omega_{1T}\right] \Rightarrow N\left(-\frac{E\left(c_i^2\theta_i^2\right)}{90}, \frac{E\left(c_i^4\right)}{45}\right)$

(b) $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[\frac{1}{T^2\sigma_i^2} \sum_{t=1}^{T} y_{it}^2 - \frac{1}{T^3\sigma_i^2} \sum_{t=1}^{T} \sum_{s=1}^{T} y_{it} y_{is} h_T(t, s) - \omega_{2T}\right] \Rightarrow N\left(-\frac{E\left(\theta_i^2\right)}{420}, \frac{11}{6300}\right)$.

Proof:

Part (a): For notational simplicity let $\bar{Y}_{it,T} = (y_{it} - y_{i0}) - \frac{t}{T}(y_{iT} - y_{i0})$ and $\bar{Y}_{it,T}(0) = (y_{it}(0) - y_{i0}(0)) - \frac{t}{T}(y_{iT}(0) - y_{i0}(0))$. Using this notation, we decompose

$$\begin{aligned}
& \frac{1}{\sqrt{n}}\sum_{i=1}^{n} c_i^2 \left[\frac{1}{T^2\sigma_i^2}\sum_{t=1}^{T}\left\{ (y_{it}-y_{i0}) - \frac{t}{T}(y_{iT}-y_{i0}) \right\}^2 - \omega_{1T} \right] \\
= & \frac{1}{\sqrt{n}}\sum_{i=1}^{n} c_i^2 \left(\frac{1}{T^2\sigma_i^2}\sum_{t=1}^{T} \bar{Y}_{it,T}^2(0) - \omega_{1T} \right) + \frac{1}{\sqrt{n}}\sum_{i=1}^{n} c_i^2 \left(\frac{1}{T^2\sigma_i^2}\sum_{t=1}^{T} \left(\bar{Y}_{it,T} - \bar{Y}_{it,T}(0)\right)^2 \right) \\
& + \frac{2}{\sqrt{n}}\sum_{i=1}^{n} c_i^2 \left(\frac{1}{T^2\sigma_i^2}\sum_{t=1}^{T} \bar{Y}_{it,T}(0)\left(\bar{Y}_{it,T} - \bar{Y}_{it,T}(0)\right) \right) \\
= & I_a + II_a + III_a, \text{ say.}
\end{aligned}$$

By a direct calculation

$$E\left[c_i^2 \left(\frac{1}{T^2\sigma_i^2}\sum_{t=1}^{T} \bar{Y}_{it,T}^2(0) - \omega_{1T} \right) \right] = O\left(\frac{1}{T}\right).$$

Also, it is possible to show that

$$\frac{1}{n}\sum_{i=1}^{n} Var\left[c_i^2 \left(\frac{1}{T^2\sigma_i^2}\sum_{t=1}^{T} \bar{Y}_{it,T}^2(0) - \omega_{1T} \right) \right] \to \frac{1}{45}E\left(c_i^4\right),$$

and

$$\sup_{i,T} E\left[c_i^2 \left(\frac{1}{T^2\sigma_i^2}\sum_{t=1}^{T} \bar{Y}_{it,T}^2(0) - \omega_{1T} \right) \right]^4 < M.$$

Then, since $\frac{n}{T} \to 0$, by the double-indexed CLT in Phillips and Moon (1999a), we have

$$I_a \Rightarrow N\left(0, \frac{1}{45}E\left(c_i^4\right)\right). \tag{1}$$

For term II_a, by definition we have

$$\begin{aligned}
& II_a \\
= & \frac{1}{\sqrt{n}}\sum_{i=1}^{n} c_i^2 \left(\frac{1}{T^2\sigma_i^2}\sum_{t=1}^{T} (y_{it} - y_{it}(0))^2 \right) \\
& - \frac{2}{\sqrt{n}}\sum_{i=1}^{n} c_i^2 \left(\frac{1}{T^2\sigma_i^2}\sum_{t=1}^{T} \left(\frac{t}{T}\right) (y_{it} - y_{it}(0))(y_{iT} - y_{iT}(0)) \right) \\
& + \frac{1}{\sqrt{n}}\sum_{i=1}^{n} c_i^2 \left(\frac{1}{T^2\sigma_i^2}\sum_{t=1}^{T} \left(\frac{t}{T}\right)^2 (y_{iT} - y_{iT}(0))^2 \right) \\
= & II_{a1} + II_{a2} + II_{a3}, \text{ say.}
\end{aligned}$$

By definition again

$$\begin{aligned}
y_{it} - y_{it}(0) &= \sum_{p=0}^{t-1}\left(\rho_i^{t-p} - 1\right) u_{ip} = \sum_{p=0}^{t-1}\left[\sum_{l=1}^{t-p} \binom{t-p}{l} \left(\frac{-\theta_i}{n^{\kappa}T}\right)^l \right] u_{ip} \text{ for } t \geq 1 \\
&= 0 \text{ for } t = 0.
\end{aligned} \tag{2}$$

Recall that $\kappa = \frac{1}{4}$. By (2) and by the WLLN, we have

$$\begin{aligned} II_{a1} &\sim \frac{1}{\sqrt{n}}\sum_{i=1}^{n} c_i^2\theta_i^2 \frac{1}{T\sigma_i^2}\sum_{t=1}^{T}\left(\frac{1}{\sqrt{T}}\sum_{p=0}^{t-1}\left(\frac{t-p}{T}\right)u_{ip}\right)^2 \\ &\to_p E\left(c_i^2\theta_i^2\right)\int_0^1\int_0^r (r-s)^2\, dsdr = \frac{1}{12}E\left(c_i^2\theta_i^2\right), \end{aligned}$$

$$\begin{aligned} II_{a2} &\sim -\frac{2}{\sqrt{n}}\sum_{i=1}^{n} c_i^2\theta_i^2\left(\frac{1}{T\sigma_i^2}\sum_{t=1}^{T}\left(\frac{t-1}{T}\right)\left(\frac{1}{\sqrt{T}}\sum_{p=0}^{t-1}\left(\frac{t-p}{T}\right)u_{ip}\right)\left(\frac{1}{\sqrt{T}}\sum_{q=0}^{T-1}\left(\frac{T-q}{T}\right)u_{iq}\right)\right) \\ &\to_p -2E\left(c_i^2\theta_i^2\right)\int_0^1 r\int_0^r (r-s)(1-s)\, dsdr = -\frac{11}{60}E\left(c_i^2\theta_i^2\right), \end{aligned}$$

and

$$\begin{aligned} II_{a3} &\sim \frac{1}{\sqrt{n}}\sum_{i=1}^{n} c_i^2\theta_i^2 \frac{1}{T\sigma_i^2}\sum_{t=1}^{T}\left(\frac{t}{T}\right)^2\left(\frac{1}{\sqrt{T}}\sum_{p=0}^{T-1}\left(\frac{T-p}{T}\right)u_{ip}\right)^2 \\ &\to_p E\left(c_i^2\theta_i^2\right)\int_0^1 r^2 dr\int_0^1 (1-r)^2\, dr = \frac{1}{9}E\left(c_i^2\theta_i^2\right). \end{aligned}$$

Combining the limits of II_{a1}, II_{a2}, and II_{a3}, we have

$$II_a \to_p \frac{1}{90}E\left(c_i^2\theta_i^2\right). \quad (3)$$

Next, for III_a, write $X_{iT} = \frac{1}{T^2\sigma^2}\sum_{t=1}^{T}\bar{Y}_{it,T}(0)\left(\bar{Y}_{it,T} - \bar{Y}_{it,T}(0)\right)$. Also define

$$X_{1iT} = \frac{1}{T\sigma_i^2}\sum_{t=1}^{T}\left[\begin{array}{c} \left(\frac{1}{\sqrt{T}}\sum_{p=0}^{t} u_{ip}\right)\left(\frac{1}{\sqrt{T}}\sum_{q=0}^{t-1}\left(\frac{t-q}{T}\right)u_{iq}\right) \\ -\left(\frac{t}{T}\right)\left(\frac{1}{\sqrt{T}}\sum_{p=0}^{t} u_{ip}\right)\left(\frac{1}{\sqrt{T}}\sum_{q=0}^{T-1}\left(\frac{T-q}{T}\right)u_{iq}\right) \\ -\left(\frac{t}{T}\right)\left(\frac{1}{\sqrt{T}}\sum_{p=0}^{T} u_{ip}\right)\left(\frac{1}{\sqrt{T}}\sum_{q=0}^{t-1}\left(\frac{t-q}{T}\right)u_{iq}\right) \\ +\left(\frac{t}{T}\right)^2\left(\frac{1}{\sqrt{T}}\sum_{p=0}^{T} u_{ip}\right)\left(\frac{1}{\sqrt{T}}\sum_{q=0}^{T-1}\left(\frac{T-q}{T}\right)u_{iq}\right) \end{array}\right],$$

and

$$X_{2iT} = \frac{1}{T\sigma_i^2}\sum_{t=1}^{T}\left[\begin{array}{c} \left(\frac{1}{\sqrt{T}}\sum_{p=0}^{t} u_{ip}\right)\left(\frac{1}{\sqrt{T}}\sum_{q=0}^{t-2}\left(\frac{t-q}{T}\right)\left(\frac{t-q-1}{T}\right)u_{iq}\right) \\ -\left(\frac{t}{T}\right)\left(\frac{1}{\sqrt{T}}\sum_{p=0}^{t} u_{ip}\right)\left(\frac{1}{\sqrt{T}}\sum_{q=0}^{T-2}\left(\frac{T-q}{T}\right)\left(\frac{T-q-1}{T}\right)u_{iq}\right) \\ -\left(\frac{t}{T}\right)\left(\frac{1}{\sqrt{T}}\sum_{p=0}^{T} u_{ip}\right)\left(\frac{1}{\sqrt{T}}\sum_{q=0}^{t-2}\left(\frac{t-q}{T}\right)\left(\frac{t-q-1}{T}\right)u_{iq}\right) \\ +\left(\frac{t}{T}\right)^2\left(\frac{1}{\sqrt{T}}\sum_{p=0}^{T} u_{ip}\right)\left(\frac{1}{\sqrt{T}}\sum_{q=0}^{T-2}\left(\frac{T-q}{T}\right)\left(\frac{T-q-1}{T}\right)u_{iq}\right) \end{array}\right].$$

Then, by (2), we have

$$III_a \sim -\frac{2}{n^{3/4}}\sum_{i=1}^{n} c_i^2\theta_i X_{1iT} + \frac{1}{n}\sum_{i=1}^{n} c_i^2\theta_i^2 X_{2iT} = -2III_{a1} + III_{a2}, \text{ say.}$$

A direct calculation shows that

$$
\begin{aligned}
& EIII_{a1} \\
= \; & \frac{E\left(c_i^2\theta_i\right)}{n^{3/4}} \sum_{i=1}^{n} EX_{1iT} \\
= \; & E\left(c_i^2\theta_i\right) n^{1/4} \frac{1}{T} \sum_{t=1}^{T} \left[\begin{array}{c} \frac{1}{T}\sum_{p=0}^{t-1} \frac{t-p}{T} - \left(\frac{t}{T}\right) \frac{1}{T} \sum_{p=0}^{t-1} \left(\frac{T-p}{T}\right) \\ -\left(\frac{t}{T}\right) \frac{1}{T} \sum_{p=0}^{t-1} \left(\frac{t-p}{T}\right) + \left(\frac{t}{T}\right)^2 \frac{1}{T} \sum_{p=0}^{T-1} \left(\frac{T-p}{T}\right) \end{array} \right] \\
= \; & E\left(c_i^2\theta_i\right) n^{1/4} \int_0^1 \left(\int_0^r (r-s)\, ds - r \int_0^r (1-s)\, ds - r \int_0^r (r-s)\, ds + r^2 \int_0^1 (1-s)\, ds \right) dr \\
& + O\left(\frac{n^{1/4}}{T}\right) \\
= \; & o(1),
\end{aligned}
$$

since $\int_0^1 \left(\int_0^r (r-s)\, ds - r \int_0^r (1-s)\, ds - r \int_0^r (r-s)\, ds + r^2 \int_0^1 (1-s)\, ds \right) dr =$ 0 and $\frac{n}{T} \to 0$. Also,

$$
\begin{aligned}
& E\left(c_i^4\theta_i^2 X_{1iT}^2\right) \\
\leq \; & \frac{2E\left(c_i^4\theta_i^2\right)}{T\sigma_i^2} \sum_{t=1}^{T} \left\{ \begin{array}{c} E\left[\left(\frac{1}{\sqrt{T}}\sum_{p=0}^{t} u_{ip}\right)\left(\frac{1}{\sqrt{T}}\sum_{q=0}^{t-1}\left(\frac{t-q}{T}\right)u_{iq}\right)\right]^2 \\ +E\left[\left(\frac{t}{T}\right)\left(\frac{1}{\sqrt{T}}\sum_{p=0}^{t} u_{ip}\right)\left(\frac{1}{\sqrt{T}}\sum_{q=0}^{T-1}\left(\frac{T-q}{T}\right)u_{iq}\right)\right]^2 \\ +E\left[\left(\frac{t}{T}\right)\left(\frac{1}{\sqrt{T}}\sum_{p=0}^{T} u_{ip}\right)\left(\frac{1}{\sqrt{T}}\sum_{q=0}^{t-1}\left(\frac{t-q}{T}\right)u_{iq}\right)\right]^2 \\ +E\left[\left(\frac{t}{T}\right)^2\left(\frac{1}{\sqrt{T}}\sum_{p=0}^{T} u_{ip}\right)\left(\frac{1}{\sqrt{T}}\sum_{q=0}^{T-1}\left(\frac{T-q}{T}\right)u_{iq}\right)\right]^2 \end{array} \right\} \\
= \; & M \text{ for some finite constant } M.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
E\left(III_{a1}^2\right) &= Var\left(III_{a1}\right) + \left(E\left(III_{a1}\right)\right)^2 \\
&\leq \frac{1}{n\sqrt{n}} \sum_{i=1}^{n} E\left(c_i^2\theta_i\right) E\left(X_{i1T}^2\right) + \left(EI_3\right)^2 \\
&= O\left(\frac{1}{n^{1/2}}\right) + O\left(\frac{n^{1/2}}{T^2}\right) = o(1),
\end{aligned}
$$

which yields

$$III_{a1} = o_p(1).$$

Next, by the WLLN, we have

$$
\begin{aligned}
III_{a2} &\to_p E\left(c_i^2\theta_i^2\right) \left[\begin{array}{c} \int_0^1 \int_0^r (r-s)^2\, ds dr - \int_0^1 r \int_0^r (1-s)^2\, ds dr \\ -\int_0^1 r \int_0^r (r-s)^2\, ds dr + \int_0^1 r^2 dr \left(\int_0^1 (1-s)^2\, ds\right) \end{array} \right] \\
&= -\frac{1}{45} E\left(c_i^2\theta_i^2\right).
\end{aligned}
$$

Combining the limits of III_{a1} and III_{a2}, we have

$$III_a \to_p -\frac{1}{45}E\left(c_i^2\theta_i^2\right). \tag{4}$$

Using (1), (3), and (4), we deduce the required result for Part (a). ■

Part (b): Write

$$\begin{aligned}&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{T^2\sigma_i^2}\sum_{t=1}^{T}y_{it}^2-\frac{1}{T^3\sigma_i^2}\sum_{t=1}^{T}\sum_{s=1}^{T}y_{it}y_{is}h_T(t,s)-\omega_{2T}\right]\\ = \;& I_b+II_b,\text{ say.}\end{aligned}$$

Rewriting term I_b as

$$I_b=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left\{\frac{1}{T^2\sigma_i^2}\sum_{t=1}^{T}y_{it}(0)^2-\frac{1}{T^3\sigma_i^2}\sum_{t=1}^{T}\sum_{s=1}^{T}y_{it}(0)\,y_{is}(0)\,h_T(t,s)-\omega_{2T}\right\},$$

and noticing that under the assumptions in the lemma,

$$E\left(\frac{1}{T^2\sigma_i^2}\sum_{t=1}^{T}y_{it}(0)^2-\frac{1}{T^3\sigma_i^2}\sum_{t=1}^{T}\sum_{s=1}^{T}y_{it}(0)\,y_{is}(0)\,h_T(t,s)\right)=\omega_{2T},$$

$$\frac{1}{n}\sum_{i=1}^{n}Var\left[\frac{1}{T^2\sigma_i^2}\sum_{t=1}^{T}y_{it}(0)^2-\frac{1}{T^3\sigma_i^2}\sum_{t=1}^{T}\sum_{s=1}^{T}y_{it}(0)\,y_{is}(0)\,h_T(t,s)-\omega_{2T}\right]\to\frac{11}{6300},$$

and

$$\sup_{i,T}E\left[\frac{1}{T^2\sigma_i^2}\sum_{t=1}^{T}y_{it}(0)^2-\frac{1}{T^3\sigma_i^2}\sum_{t=1}^{T}\sum_{s=1}^{T}y_{it}(0)\,y_{is}(0)\,h_T(t,s)-\omega_{2T}\right]^4<M,$$

we apply the double-indexed CLT in Phillips and Moon (1999a) and deduce that

$$I_b \Rightarrow N\left(0,\frac{11}{6300}\right). \tag{5}$$

For II_b, we further decompose the term II_b into the components

$$II_b = II_{b1}+2II_{b2},$$

where

$$II_{b1}=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{T^2\sigma_i^2}\sum_{t=1}^{T}\left(y_{it}-y_{it}(0)\right)^2-\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{T^3\sigma_i^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\left(y_{it}-y_{it}(0)\right)\left(y_{is}-y_{is}(0)\right)h_T(t,s)$$

and

$$II_{b2}=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{T^2\sigma_i^2}\sum_{t=1}^{T}\left(y_{it}-y_{it}(0)\right)y_{it}(0)-\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{T^3\sigma_i^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\left(y_{it}-y_{it}(0)\right)y_{is}(0)\,h_T(t,s).$$

For II_{b1}, by (2) and by the WLLN, we have

$$\begin{aligned}
&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{T^2\sigma_i^2}\sum_{t=1}^{T}\left(y_{it}-y_{it}\left(0\right)\right)^2 \\
= &\ \frac{1}{n}\sum_{i=1}^{n}\theta_i^2\left[\frac{1}{T^2\sigma_i^2}\sum_{t=1}^{T}\left(\sum_{p=0}^{t-1}\left(\frac{t-p}{T}\right)u_{ip}\right)^2\right]+O_p\left(\frac{1}{n^{1/4}}\right) \\
\to_p &\ E\left(\theta_i^2\right)\int_0^1\int_0^r\left(r-s\right)^2 dsdr=\frac{1}{12}E\left(\theta_i^2\right),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{T^3\sigma_i^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\left(y_{it}-y_{it}\left(0\right)\right)\left(y_{is}-y_{is}\left(0\right)\right)h_T\left(t,s\right) \\
= &\ \frac{1}{n}\sum_{i=1}^{n}\theta_i^2\frac{1}{T^3\sigma_i^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{p=0}^{t-1}\sum_{q=0}^{s-1}\left(\frac{t-p}{T}\right)\left(\frac{s-q}{T}\right)h_T\left(t,s\right)u_{ip}u_{iq} \\
&+O_p\left(\frac{1}{n^{1/4}}\right) \\
\to_p &\ E\left(\theta_i^2\right)\int_0^1\int_0^1\int_0^{r\wedge s}\left(r-p\right)\left(s-p\right)h\left(r,s\right)dpdsdr=\frac{17}{210}E\left(\theta_i^2\right).
\end{aligned}$$

Therefore,

$$II_{b1}\to_p\frac{1}{420}E\left(\theta_i^2\right). \tag{6}$$

Next, in view of (2) with $\kappa=\frac{1}{4}$, we have

$$II_{b2}=-\frac{2}{n^{3/4}}\sum_{i=1}^{n}\theta_i X_{1iT}+\frac{1}{n}\sum_{i=1}^{n}\theta_i^2 X_{2iT}+o_p\left(1\right),$$

where

$$X_{1iT}=\frac{1}{T^2\sigma_i^2}\sum_{t=1}^{T}\sum_{s=0}^{t-1}\sum_{q=0}^{t}\left(\frac{t-s}{T}\right)u_{is}u_{iq}-\frac{1}{T^3\sigma_i^2}\sum_{t=1}^{T}\sum_{p=1}^{T}\sum_{s=0}^{t-1}\sum_{q=0}^{p}\left(\frac{t-s}{T}\right)h_T\left(t,p\right)u_{is}u_{iq},$$

and

$$\begin{aligned}
X_{2iT} = &\ \frac{1}{T^2\sigma_i^2}\sum_{t=1}^{T}\sum_{s=0}^{t-1}\sum_{q=0}^{t}\left(\frac{t-s}{T}\right)\left(\frac{t-s-1}{T}\right)u_{is}u_{iq} \\
&-\frac{1}{T^3\sigma_i^2}\sum_{t=1}^{T}\sum_{p=1}^{T}\sum_{s=0}^{t-1}\sum_{q=0}^{p}\left(\frac{t-s}{T}\right)\left(\frac{t-s-1}{T}\right)h_T\left(t,p\right)u_{is}u_{iq}.
\end{aligned}$$

A direct calculation shows that

$$EX_{1iT}=\left[\frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{t-1}\left(\frac{t-s}{T}\right)-\frac{1}{T^3}\sum_{t=1}^{T}\sum_{p=1}^{T}\sum_{s=0}^{t\wedge p-1}\left(\frac{t-s}{T}\right)h_T\left(t,p\right)\right]=O\left(\frac{1}{T}\right),$$

because

$$EX_{1iT}-\int_0^1\int_0^r(r-s)\,dsdr+\sigma^2\int_0^1\int_0^1\int_0^{r\wedge p}(r-s)\,h(r,p)\,dsdpdr=O\left(\frac{1}{T}\right),$$

and

$$\int_0^1\int_0^r(r-s)\,dsdr-\int_0^1\int_0^1\int_0^{r\wedge p}(r-s)\,h(r,p)\,dsdpdr=0.$$

Also,

$$\begin{aligned}
&\sup_i EX_{1iT}^2\\
\leq\;&\sup_i\frac{2}{T^4\sigma_i^4}\sum_{t=1}^T\sum_{x=1}^T\sum_{s=0}^{t-1}\sum_{y=0}^{x-1}\sum_{q=0}^{t}\sum_{z=0}^{x}\left(\frac{t-s}{T}\right)\left(\frac{x-y}{T}\right)E\left[u_{is}u_{iq}u_{iy}u_{iz}\right]\\
&+\sup_i\frac{2}{T^6\sigma_i^4}\sum_{t=1}^T\sum_{p=1}^T\sum_{x=1}^T\sum_{y=1}^T\sum_{s=0}^{t-1}\sum_{q=0}^{p-1}\sum_{z=0}^{x}\sum_{w=0}^{y}\left(\frac{t-s}{T}\right)\left(\frac{x-z}{T}\right)h_T(t,p)\,h_T(x,y)\,E\left[u_{is}u_{iq}u_{iz}u_{iw}\right]\\
=\;&O(1).
\end{aligned}$$

Therefore

$$\begin{aligned}
-\frac{1}{n^{3/4}}\sum_{i=1}^n\theta_iX_{1iT} &= -\frac{1}{n^{3/4}}\sum_{i=1}^n\theta_i\left(X_{1iT}-EX_{1iT}\right)+\frac{1}{n^{3/4}}\sum_{i=1}^n\theta_i\left(EX_{1iT}\right)\\
&= O_p\left(\frac{1}{n^{1/4}}\right)+O\left(\frac{n^{1/4}}{T}\right)=o_p(1).
\end{aligned}$$

Next, by the WLLN, we have

$$\begin{aligned}
\frac{1}{n\sigma^2}\sum_{i=1}^n\theta_i^2X_{2iT} &\to_p E\left(\theta_i^2\right)\left[\int_0^1\int_0^r(r-s)^2\,dsdr-\int_0^1\int_0^1\int_0^{r\wedge p}(r-s)^2\,h(r,p)\,dsdpdr\right]\\
&= -E\left(\theta_i^2\right)\frac{1}{210}.
\end{aligned}$$

Therefore, we have

$$II_{b2}\to_p -E\left(\theta_i^2\right)\frac{1}{210}. \quad (7)$$

Combining the limits of the terms I_b, II_{b1}, and II_{b2} in (5), (6), and (7), respectively, we have the desired result for Part (b). ■

Lemma 7 *Let M be a finite constant. Under Assumptions 1 and 2, the following hold.*

(a) $\sup_i E\left[\left(\frac{1}{T}\sum_{t=1}^T u_{it}y_{it-1}\right)^2\right]<M.$

(b) $\sup_i E\left[\left(\frac{1}{T^2}\sum_{t=1}^{T} y_{it-1}^2\right)^2\right] < M.$

(c) $\sup_i E\left[y_{i0}^2\right] < M.$

Proof. The lemma follows by direct calculation and we omit the proof. ■

Lemma 8 *Suppose that Assumptions 1–3 and 4 hold. Then, the following hold.*

(a) $\sum_{i=1}^{n}\left(\hat{\sigma}_{1,iT}^2 - \sigma_i^2\right)^2 = o_p(1)$.

(b) $\sup_{1\le i\le n}\left|\hat{\sigma}_{1,iT}^2 - \sigma_i^2\right| = o_p(1)$.

(c) *With probability approaching one, there exists a constant $M > 0$ such that* $\inf_i \hat{\sigma}_{1,iT}^2 \ge M$.

Proof.
Part (a): The required result follows by Lemma 5(a) if $\sum_{i=1}^{n}\left(\hat{\sigma}_{1,iT}^2 - \tilde{\sigma}_{iT}^2\right)^2 = o_p(1)$. Under Assumption 4, we have

$$\hat{\sigma}_{1,iT}^2 - \tilde{\sigma}_{iT}^2 = \frac{y_{i0}^2}{T} + \frac{\theta_i^2}{nT}\left(\frac{1}{T^2}\sum_{t=1}^{T} y_{it-1}^2\right) - 2\frac{\theta_i}{n^{1/2}T}\left(\frac{1}{T}\sum_{t=1}^{T} u_{it}y_{it-1}\right).$$

Then, by Lemma 7 and since the support of θ_i is uniformly bounded, we have for some constant M,

$$\begin{aligned}
&\sum_{i=1}^{n} E\left(\hat{\sigma}_{1,iT}^2 - \tilde{\sigma}_{iT}^2\right)^2 \\
\le\ & M\left[\frac{1}{T^2}\sum_{i=1}^{n} E\left(y_{i0}^4\right) + \frac{1}{n^2T^2}\sum_{i=1}^{n} E\left(\frac{1}{T^2}\sum_{t=1}^{T} y_{it-1}^2\right)^2 + \frac{1}{nT^2}\sum_{i=1}^{n}\left(\frac{1}{T}\sum_{t=1}^{T} u_{it}y_{it-1}\right)^2\right] \\
=\ & O\left(\frac{n}{T^2}\right) + O\left(\frac{1}{nT^2}\right) + O\left(\frac{1}{T^2}\right) = o(1), \qquad (8)
\end{aligned}$$

as required. ■
Parts (b) and (c): Since $\inf_{1\le i\le n}\hat{\sigma}_{1,iT}^2 \le \inf_i \sigma_i^2 - \sup_{1\le i\le n}\left|\hat{\sigma}_{1,iT}^2 - \sigma_i^2\right|$, under Assumption 1, the required results follow if we show that

$$\sup_{1\le i\le n}\left|\hat{\sigma}_{1,iT}^2 - \sigma_i^2\right| = o_p(1),$$

which follows by Lemma 5(b) if we show that

$$\sup_{1\le i\le n}\left|\hat{\sigma}_{1,iT}^2 - \tilde{\sigma}_{iT}^2\right| = o_p(1).$$

This follows since for any $\varepsilon > 0$,

$$P\left\{\sup_{1\leq i\leq n}\left|\hat{\sigma}_{1,iT}^2 - \tilde{\sigma}_{iT}^2\right| > \varepsilon\right\} \leq \sum_{i=1}^{n} P\left\{\left|\hat{\sigma}_{1,iT}^2 - \tilde{\sigma}_{iT}^2\right| > \varepsilon\right\} \leq \frac{1}{\varepsilon^2}\sum_{i=1}^{n} E\left[\left(\hat{\sigma}_{1,iT}^2 - \tilde{\sigma}_{iT}^2\right)^2\right] \to 0$$

by (8) ■

Lemma 9 *Let M be a finite constant. Under Assumptions 1 and 2, the following hold.*

(a) $\sup_i E\left[\left(\frac{y_{iT}-y_{i0}}{\sqrt{T}}\right)^2\right] < M.$

(b) $\sup_i E\left[\left(\frac{1}{T\sqrt{T}}\sum_{t=1}^{T} y_{it-1}\right)^2\right] < M.$

Proof. The lemma follows by direct calculation, and its proof is omitted. ■

Lemma 10 *Suppose that Assumptions 1 – 3, and 4 hold. Then, the following hold.*

(a) $\sup_{1\leq i\leq n}\left(\hat{\sigma}_{2,iT}^2 - \sigma_i^2\right) = o_p(1)$.

(b) *With probability approaching one, there exists a constant $M > 0$ such that $\inf_i \hat{\sigma}_{2,iT}^2 \geq M$.*

Proof.
Part (a): The required result follows by Lemmas 5(b) and 8(b) and the triangle inequality, if we show that $\sup_{1\leq i\leq n}\left|\hat{\sigma}_{2,iT}^2 - \hat{\sigma}_{1,iT}^2\right| = o_p(1)$ because $\hat{\sigma}_{1,iT}^2 = \frac{1}{T}\sum_{t=1}^{T}(\Delta y_{it})^2$. By definition, $\sup_{1\leq i\leq n}\left|\hat{\sigma}_{2,iT}^2 - \hat{\sigma}_{1,iT}^2\right| = \sup_{1\leq i\leq n}\frac{y_{i0}^2}{T}$. Then, by the Markov's inequality, for any $\varepsilon > 0$,

$$P\left\{\sup_{1\leq i\leq n}\frac{y_{i0}^2}{T} > \varepsilon\right\} \leq \sum_{i=1}^{n} P\left\{\frac{y_{i0}^2}{T} > \varepsilon\right\} \leq \frac{1}{\varepsilon T}\sum_{i=1}^{n} E y_{i0}^2 = O\left(\frac{n}{T}\right) = o(1).\ ■$$

Part (b): Since $\inf_{1\leq i\leq n}\hat{\sigma}_{2,iT}^2 \leq \inf_i \sigma_i^2 - \sup_{1\leq i\leq n}\left|\hat{\sigma}_{2,iT}^2 - \sigma_i^2\right|$ under Assumption 1, the required result follows by Part (a). ■

Lemma 11 *Under Assumptions 1–3, 6, and 7,*

$$
\begin{aligned}
& V_{fe2,nT}\left(\mathbb{C}\right) \\
= \; & \frac{1}{n^{1/4}}\sum_{i=1}^{n}\frac{c_i}{\sigma_i^2}\left[\frac{2}{T}\sum_{t=1}^{T}\Delta y_{it}y_{it-1}-\left(\frac{y_{iT}}{\sqrt{T}}\right)^2+\left(\frac{y_{i0}}{\sqrt{T}}\right)^2+\sigma_i^2\right] \\
& +\frac{1}{n^{1/2}}\sum_{i=1}^{n}\frac{c_i^2}{\sigma_i^2}\left[\begin{array}{c}\frac{1}{T^2}\sum_{t=1}^{T}y_{it-1}^2-2\left(\frac{y_{iT}}{\sqrt{T}}\right)\left(\frac{1}{T\sqrt{T}}\sum_{t=1}^{T}\frac{t}{T}y_{it-1}\right) \\ +\frac{1}{3}\left(\frac{y_{iT}}{\sqrt{T}}\right)^2+\sigma_i^2\omega_{p2T}\end{array}\right] \\
& +\frac{1}{n}\sum_{i=1}^{n}\frac{c_i^4}{\sigma_i^2}\left[\begin{array}{c}-\left(\frac{1}{T\sqrt{T}}\sum_{t=1}^{T}\frac{t}{T}y_{it-1}\right)^2+\frac{2}{3}\left(\frac{y_{iT}}{\sqrt{T}}\right)\left(\frac{1}{T\sqrt{T}}\sum_{t=1}^{T}\frac{t}{T}y_{it-1}\right) \\ -\frac{1}{9}\left(\frac{y_{iT}}{\sqrt{T}}\right)^2+\sigma_i^2\omega_{p4T}\end{array}\right] \\
& +\frac{1}{n^{1/4}T}\sum_{i=1}^{n}\frac{\mathcal{S}_{1iT}}{\sigma_i^2}+\frac{1}{n^{1/2}T^{1/2}}\sum_{i=1}^{n}\frac{\mathcal{S}_{2iT}}{\sigma_i^2}+\frac{1}{n^{5/4}}\sum_{i=1}^{n}\frac{\mathcal{S}_{3iT}}{\sigma_i^2},
\end{aligned}
$$

with $\frac{1}{n}\sum_{i=1}^{n}E\left[\mathcal{S}_{kiT}^2\right]=O\left(1\right)$, *for* $k=1,2,3$ *when* $\left(n,T\rightarrow\infty\right)$ *with* $\frac{n}{T}\rightarrow 0$.

Proof: By definition,

$$
\begin{aligned}
V_{fe2,nT}\left(\mathbb{C}\right) = \; & V_{fe21,nT}\left(\mathbb{C}\right)+V_{fe22,nT}\left(\mathbb{C}\right) \\
& +\left(\frac{1}{n^{1/4}}\sum_{i=1}^{n}c_i\right)+\left(\frac{1}{n^{1/2}}\sum_{i=1}^{n}c_i^2\right)\left(-\frac{1}{T}\sum_{t=2}^{T}\frac{t-1}{T}+2\frac{1}{T}\sum_{t=2}^{T}\left(\frac{t-1}{T}\right)^2-\frac{1}{3}\right) \\
& +\left(\frac{1}{n}\sum_{i=1}^{n}c_i^4\right)\left(\frac{1}{T^2}\sum_{t=2}^{T}\sum_{s=2}^{T}\frac{t-1}{T}\frac{s-1}{T}\min\left(\frac{t-1}{T},\frac{s-1}{T}\right)-\frac{2}{3}\left(\frac{1}{T}\sum_{t=2}^{T}\left(\frac{t-1}{T}\right)^2\right)+\frac{1}{9}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
V_{fe21,nT}\left(\mathbb{C}\right) & = \sum_{i=1}^{n}\frac{1}{\sigma_i^2}\left[\left(\Delta_{c_i}\underline{Y}_i\right)'\left(\Delta_{c_i}\underline{Y}_i\right)-\left(\Delta\underline{Y}_i\right)'\left(\Delta\underline{Y}_i\right)\right] \\
& = \frac{2}{n^{1/4}}\sum_{i=1}^{n}\frac{c_i}{\sigma_i^2}\left(\frac{1}{T}\sum_{t=1}^{T}\Delta y_{it}y_{it-1}\right)+\frac{1}{n^{1/2}T^2}\sum_{i=1}^{T}\frac{c_i^2}{\sigma_i^2}\sum_{t=1}^{T}y_{it-1}^2 \\
V_{fe22,nT}\left(\mathbb{C}\right) & = \sum_{i=1}^{n}\frac{1}{\sigma_i^2}\left[\begin{array}{c}\left(\Delta\underline{Y}_i\right)'\Delta G\left(\Delta G'\Delta G\right)^{-1}\Delta G'\left(\Delta\underline{Y}_i\right) \\ -\left(\Delta_{c_i}\underline{Y}_i\right)'\Delta_{c_i}G\left(\Delta_{c_i}G'\Delta_{c_i}G\right)^{-1}\Delta_{c_i}G'\left(\Delta_{c_i}\underline{Y}_i\right)\end{array}\right].
\end{aligned}
$$

Let $D = diag(\sqrt{T}, 1)$ and $\tilde{G} = GD$. Then,

$$
\begin{aligned}
& V_{fe22,nT}(\mathbb{C}) \\
= & \sum_{i=1}^{n} \frac{1}{\sigma_i^2} \left(\frac{1}{\sqrt{T}} (\Delta \underline{Y}_i)' \Delta \tilde{G} \right) \left(\frac{1}{T} \Delta \tilde{G}' \Delta \tilde{G} \right)^{-1} \left(\frac{1}{\sqrt{T}} \Delta \tilde{G}' (\Delta \underline{Y}_i) \right) \\
& - \sum_{i=1}^{n} \frac{1}{\sigma_i^2} \left(\frac{1}{\sqrt{T}} (\Delta_{c_i} \underline{Y}_i)' \Delta_{c_i} \tilde{G} \right) \left(\frac{1}{T} \Delta_{c_i} \tilde{G}' \Delta_{c_i} \tilde{G} \right)^{-1} \left(\frac{1}{\sqrt{T}} \Delta_{c_i} \tilde{G}' (\Delta_{c_i} \underline{Y}_i) \right) \\
= & \sum_{i=1}^{n} \frac{1}{\sigma_i^2} tr \left[\begin{array}{c} \left(\frac{1}{T} \Delta \tilde{G}' \Delta \tilde{G} \right)^{-1} \\ \times \left\{ \left(\frac{1}{\sqrt{T}} \Delta \tilde{G}' (\Delta \underline{Y}_i) \right) \left(\frac{1}{\sqrt{T}} (\Delta \underline{Y}_i)' \Delta \tilde{G} \right)' - \left(\frac{1}{\sqrt{T}} \Delta_{c_i} \tilde{G}' (\Delta_{c_i} \underline{Y}_i) \right) \left(\frac{1}{\sqrt{T}} (\Delta_{c_i} \underline{Y}_i)' \Delta_{c_i} \tilde{G} \right)' \right\} \end{array} \right] \\
& + \sum_{i=1}^{n} \frac{1}{\sigma_i^2} tr \left[\left\{ \left(\frac{1}{T} \Delta \tilde{G}' \Delta \tilde{G} \right)^{-1} - \left(\frac{1}{T} \Delta_{c_i} \tilde{G}' \Delta_{c_i} \tilde{G} \right)^{-1} \right\} \left(\frac{1}{\sqrt{T}} \Delta_{c_i} \tilde{G}' (\Delta_{c_i} \underline{Y}_i) \right) \left(\frac{1}{\sqrt{T}} (\Delta_{c_i} \underline{Y}_i)' \Delta_{c_i} \tilde{G} \right)' \right] \\
= & V_{fe221,nT}(\mathbb{C}) + V_{fe222,nT}(\mathbb{C}), \text{ say.}
\end{aligned}
$$

Notice that

$$
\begin{aligned}
\frac{1}{T} \Delta \tilde{G}' \Delta \tilde{G} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
\frac{1}{T} \Delta_{c_i} \tilde{G}' \Delta_{c_i} \tilde{G} &= \begin{pmatrix} 1 + \frac{c_i^2}{n^{1/2}} \frac{1}{T} & \frac{1}{\sqrt{T}} \left(\frac{c_i}{n^{1/4}} + \frac{c_i^2}{n^{1/2}} \left(\frac{1}{T} \sum_{t=1}^{T} \frac{t}{T} \right) \right) \\ \frac{1}{\sqrt{T}} \left(\frac{c_i}{n^{1/4}} + \frac{c_i^2}{n^{1/2}} \left(\frac{1}{T} \sum_{t=1}^{T} \frac{t}{T} \right) \right) & \frac{1}{T} \sum_{t=1}^{T} \left(1 + \frac{c_i}{n^{1/4}} \frac{t}{T} \right)^2 \end{pmatrix}, \\
\frac{1}{\sqrt{T}} \Delta \tilde{G}' (\Delta \underline{Y}_i) &= \begin{pmatrix} y_{i0} \\ \frac{1}{\sqrt{T}} (y_{iT} - y_{i0}) \end{pmatrix}, \\
\frac{1}{\sqrt{T}} \Delta_{c_i} \tilde{G}' (\Delta_{c_i} \underline{Y}_i) &= \begin{pmatrix} y_{i0} + \frac{c_i}{n^{1/4}} \frac{1}{T} (y_{iT} - y_{i1}) + \frac{c_i^2}{n^{1/2}} \frac{1}{T^2} \sum_{t=1}^{T} y_{it-1} \\ \frac{1}{\sqrt{T}} (y_{iT} - y_{i0}) + \frac{c_i}{n^{1/4}} \frac{1}{\sqrt{T}} y_{iT} + \frac{c_i^2}{n^{1/2}} \frac{1}{T\sqrt{T}} \sum_{t=1}^{T} \frac{t}{T} y_{it-1} \end{pmatrix}.
\end{aligned}
$$

Computation of $V_{fe221,nT}(\mathbb{C})$: A direct calculation shows that

$$
\begin{aligned}
& V_{fe221,nT}(\mathbb{C}) \\
= & \frac{1}{n^{1/4}} \sum_{i=1}^{n} \frac{c_i}{\sigma_i^2} \left(2 \left(\frac{y_{i0}}{\sqrt{T}} \right)^2 - 2 \left(\frac{y_{iT}}{\sqrt{T}} \right)^2 \right) \\
& + \frac{1}{n^{1/2}} \sum_{i=1}^{n} \frac{c_i^2}{\sigma_i^2} \left(-2 \left(\frac{y_{iT}}{\sqrt{T}} \right) \left(\frac{1}{T\sqrt{T}} \sum_{t=1}^{T} \frac{t}{T} y_{it-1} \right) - \left(\frac{y_{iT}}{\sqrt{T}} \right)^2 + \frac{1}{T} \mathcal{R}_{2iT} \right) \\
& + \frac{1}{n^{3/4}} \sum_{i=1}^{n} \frac{c_i^3}{\sigma_i^2} \left(-2 \left(\frac{y_{iT}}{\sqrt{T}} \right) \left(\frac{1}{T\sqrt{T}} \sum_{t=1}^{T} \frac{t}{T} y_{it-1} \right) + \frac{1}{T} \mathcal{R}_{3iT} \right) \\
& + \frac{1}{n} \sum_{i=1}^{n} \frac{c_i^4}{\sigma_i^2} \left(- \left(\frac{1}{T\sqrt{T}} \sum_{t=1}^{T} \frac{t}{T} y_{it-1} \right)^2 + \frac{1}{T} \mathcal{R}_{4iT} \right),
\end{aligned}
$$

where $\frac{1}{n} \sum_{i=1}^{n} E \mathcal{R}_{kiT}^2 = O(1)$ for $k = 2, 3, 4$.

Computation of $V_{fe222,nT}(\mathbb{C})$:
Direct calculation gives

$$\left(\frac{1}{T}\Delta\tilde{G}'\Delta\tilde{G}\right)^{-1} - \left(\frac{1}{T}\Delta_{c_i}\tilde{G}'\Delta_{c_i}\tilde{G}\right)^{-1}$$

$$= \begin{pmatrix} 0 & \frac{1}{\sqrt{T}}\left(\frac{c_i}{n^{1/4}} - \frac{1}{2}\frac{c_i^2}{n^{1/2}} + \frac{1}{6}\frac{c_i^3}{n^{3/4}}\right) \\ \frac{1}{\sqrt{T}}\left(\frac{c_i}{n^{1/4}} - \frac{1}{2}\frac{c_i^2}{n^{1/2}} + \frac{1}{6}\frac{c_i^3}{n^{3/4}}\right) & \frac{c_i}{n^{1/4}}\left(2\left(\frac{1}{T}\sum_{t=1}^T \frac{t}{T}\right)\right) - \frac{2}{3}\frac{c_i^2}{n^{1/2}} + \frac{1}{3}\frac{c_i^3}{n^{3/4}} - \frac{1}{9}\frac{c_i^4}{n} \end{pmatrix}$$

$$+\xi_{1,i,nT} + \xi_{2,i,nT},$$

where $\xi_{1,i,nT} = O\left(\frac{1}{n^{1/2}T}\right)$ and $\xi_{2,i,nT} = O\left(\frac{1}{n^{5/4}}\right)$ uniformly across i because the support of $c_i's$ is bounded. Then,

$$\begin{aligned}
& V_{fe222,nT}(\mathbb{C}) \\
= \; & \frac{1}{n^{1/4}}\sum_{i=1}^n \frac{c_i}{\sigma_i^2}\left(\left(\frac{y_{iT}}{\sqrt{T}}\right)^2 - \left(\frac{y_{i0}}{\sqrt{T}}\right)^2\right) + \frac{1}{n^{1/4}T}\sum_{i=1}^n \frac{\mathcal{R}_{5iT}}{\sigma_i^2} \\
& + \frac{1}{n^{1/2}}\sum_{i=1}^n \frac{c_i^2}{\sigma_i^2}\frac{4}{3}\left(\frac{y_{iT}}{\sqrt{T}}\right)^2 + \frac{1}{n^{1/2}T^{1/2}}\sum_{i=1}^n \frac{\mathcal{R}_{6iT}}{\sigma_i^2} \\
& + \frac{1}{n^{3/4}}\sum_{i=1}^n \frac{c_i^3}{\sigma_i^2}2\left(\frac{y_{iT}}{\sqrt{T}}\right)\left(\frac{1}{T\sqrt{T}}\sum_{t=1}^T \frac{t}{T}y_{it-1}\right) + \frac{1}{n^{3/4}T^{1/2}}\sum_{i=1}^n \frac{\mathcal{R}_{7iT}}{\sigma_i^2} \\
& + \frac{1}{n}\sum_{i=1}^n \frac{c_i^4}{\sigma_i^2}\left(\frac{2}{3}\left(\frac{y_{iT}}{\sqrt{T}}\right)\left(\frac{1}{T\sqrt{T}}\sum_{t=1}^T \frac{t}{T}y_{it-1}\right) - \frac{1}{9}\left(\frac{y_{iT}}{\sqrt{T}}\right)^2\right) + \frac{1}{nT^{1/2}}\sum_{i=1}^n \frac{\mathcal{R}_{8iT}}{\sigma_i^2} \\
& + \frac{1}{n^{5/4}}\sum_{i=1}^n \frac{\mathcal{R}_{9iT}}{\sigma_i^2},
\end{aligned}$$

where $\frac{1}{n}\sum_{i=1}^n E\mathcal{R}_{kiT}^2 = O(1)$ for $k = 5, ..., 9$. Putting the terms in $V_{fe21,nT}(\mathbb{C})$, $V_{fe221,nT}(\mathbb{C})$, and $V_{fe222,nT}(\mathbb{C})$ together with the restriction that $n, T \to \infty$ with $\frac{n}{T} \to 0$, we have the required result. ■

Lemma 12 *Under Assumptions 1 – 3, 6, and 7, the following hold:*

(a) $\frac{1}{n^{1/4}}\sum_{i=1}^n \frac{c_i}{\sigma_i^2}\left[\frac{2}{T}\sum_{t=1}^T \Delta y_{it}y_{it-1} - \left(\frac{y_{iT}}{\sqrt{T}}\right)^2 + \left(\frac{y_{i0}}{\sqrt{T}}\right)^2 + \sigma_i^2\right] = o_p(1);$

(b) $\frac{1}{n^{1/2}}\sum_{i=1}^n \frac{c_i^2}{\sigma_i^2}\left[\begin{array}{c} \left(\frac{1}{T^2}\sum_{t=1}^T y_{it-1}^2 - \sigma_i^2\frac{1}{T}\sum_{t=1}^T \frac{t-1}{T}\right) + \frac{1}{3}\left\{\left(\frac{y_{iT}}{\sqrt{T}}\right)^2 - \sigma_i^2\right\} \\ -\left\{2\left(\frac{y_{iT}}{\sqrt{T}}\right)\left(\frac{1}{T\sqrt{T}}\sum_{t=1}^T \frac{t}{T}y_{it-1}\right) - \sigma_i^2\frac{2}{T}\sum_{t=1}^T\left(\frac{t}{T}\right)\left(\frac{t-1}{T}\right)\right\} \end{array}\right] \Rightarrow$
$N\left(-\frac{1}{90}E\left(c_i^2\theta_i^2\right), \frac{1}{45}E\left(c_i^4\right)\right);$

(c) $\frac{1}{n}\sum_{i=1}^n \frac{c_i^4}{\sigma_i^2}\left[\begin{array}{c} -\left(\frac{1}{T\sqrt{T}}\sum_{t=1}^T \frac{t}{T}y_{it-1}\right)^2 + \frac{2}{3}\left(\frac{y_{iT}}{\sqrt{T}}\right)\left(\frac{1}{T\sqrt{T}}\sum_{t=1}^T \frac{t}{T}y_{it-1}\right) \\ -\frac{1}{9}\left(\frac{y_{iT}}{\sqrt{T}}\right)^2 + \sigma_i^2\omega_{p4T} \end{array}\right] =$
$o_p(1).$

Proof:
Part (a): First, notice from

$$y_{it}^2 - y_{it-1}^2 = \left(\rho_i^2 - 1\right) y_{it-1}^2 + 2\rho_i y_{it-1} u_{it} + u_{it}^2 \text{ for } t \geq 1,$$

that

$$\left(\frac{y_{iT}}{\sqrt{T}}\right)^2 - \left(\frac{y_{i0}}{\sqrt{T}}\right)^2 = \left(\rho_i^2 - 1\right) \frac{1}{T} \sum_{t=1}^T y_{it-1}^2 + 2\rho_i \frac{1}{T} \sum_{t=1}^T y_{it-1} u_{it} + \frac{1}{T} \sum_{t=1}^T u_{it}^2.$$

Since $\Delta y_{it} = (\rho_i - 1)\, y_{it-1} + u_{it}$, we have

$$2\frac{1}{T} \sum_{t=1}^T \Delta y_{it} y_{it-1} = 2\left(\rho_i - 1\right) \frac{1}{T} \sum_{t=1}^T y_{it-1}^2 + \frac{1}{T} \sum_{t=1}^T y_{it-1} u_{it}.$$

Then,

$$\begin{aligned} & \frac{1}{n^{1/4}} \sum_{i=1}^n \frac{c_i}{\sigma_i^2} \left[\frac{2}{T} \sum_{t=1}^T \Delta y_{it} y_{it-1} - \left(\frac{y_{iT}}{\sqrt{T}}\right)^2 + \left(\frac{y_{i0}}{\sqrt{T}}\right)^2 + \sigma_i^2\right] \\ = \; & \frac{1}{n^{1/4}} \sum_{i=1}^n \frac{c_i}{\sigma_i^2} \left[-\left(\rho_i - 1\right)^2 \frac{1}{T} \sum_{t=1}^T y_{it-1}^2 + 2\left(1 - \rho_i\right) \frac{1}{T} \sum_{t=1}^T y_{it-1} u_{it} - \left(\frac{1}{T} \sum_{t=1}^T u_{it}^2 - \sigma_i^2\right)\right]. \end{aligned}$$

Under the assumptions of the lemma,

$$\frac{1}{n^{1/4}} \sum_{i=1}^n \frac{c_i}{\sigma_i^2} \left(\rho_i - 1\right)^2 \frac{1}{T} \sum_{t=1}^T y_{it-1}^2 = \frac{n^{1/4}}{T} \left(\frac{1}{n} \sum_{i=1}^n c_i \theta_i^2 \left(\frac{1}{T^2 \sigma_i^2} \sum_{t=1}^T y_{it-1}^2\right)\right) = O_p\left(\frac{n^{1/4}}{T}\right),$$

$$\frac{1}{n^{1/4}} \sum_{i=1}^n \frac{c_i}{\sigma_i^2} \left(1 - \rho_i\right) \frac{1}{T} \sum_{t=1}^T y_{it-1} u_{it} = \frac{1}{T} \frac{2}{n^{1/2}} \sum_{i=1}^n c_i \left(\frac{1}{T\sigma_i^2} \sum_{t=1}^T y_{it-1} u_{it}\right) = O_p\left(\frac{1}{T}\right),$$

and

$$\frac{1}{n^{1/4}} \sum_{i=1}^n \frac{c_i}{\sigma_i^2} \left(\frac{1}{T} \sum_{t=1}^T u_{it}^2 - \sigma_i^2\right) = O_p\left(\frac{n^{1/4}}{T^{1/2}}\right),$$

leading to the required result for Part (a). ■
Part (b): Under the assumptions of the lemma, it is possible to show that

$$\begin{aligned} & \frac{1}{n^{1/2}} \sum_{i=1}^n \frac{c_i^2}{\sigma_i^2} \left[\begin{array}{c} \left(\frac{1}{T^2} \sum_{t=1}^T y_{it-1}^2 - \sigma_i^2 \frac{1}{T} \sum_{t=1}^T \frac{t-1}{T}\right) + \frac{1}{3}\left\{\left(\frac{y_{iT}}{\sqrt{T}}\right)^2 - \sigma_i^2\right\} \\ -\left\{2\left(\frac{y_{iT}}{\sqrt{T}}\right)\left(\frac{1}{T\sqrt{T}} \sum_{t=1}^T \frac{t-1}{T} y_{it-1}\right) - \sigma_i^2 \frac{2}{T} \sum_{t=1}^T \left(\frac{t-1}{T}\right)^2\right\} \end{array}\right] \\ = \; & \frac{1}{\sqrt{n}} \sum_{i=1}^n c_i^2 \left[\frac{1}{T^2 \sigma_i^2} \sum_{t=1}^T \left\{\left(y_{it} - y_{i0}\right) - \frac{t}{T}\left(y_{iT} - y_{i0}\right)\right\}^2 - \omega_{1T}\right] + o_p(1) \\ \Rightarrow \; & N\left(-\frac{1}{90} E\left(c_i^2 \theta_i^2\right), \frac{1}{45} E\left(c_i^4\right)\right), \end{aligned}$$

where the last limit holds by Lemma 6(a). ■

Part (c): Under the assumptions of the lemma, Part (c) follows by the WLLN. ■

Lemma 13 *Let M be a finite constant. Under Assumptions 1 – 3, 6, and 7, the following hold.*

(a) $\sup_i E\left[y_{i0}^4\right] < M$.

(b) $\sup_i E\left[\left(\frac{y_{iT}}{\sqrt{T}}\right)^4\right] < M$.

(c) $\sup_i E\left[\left(\frac{1}{T}\sum_{t=1}^T y_{it-1}u_{it}\right)^2\right] < M$.

(d) $\sup_i E\left[\left(\frac{1}{T^2}\sum_{t=1}^T y_{it-1}^2\right)^2\right] < M$.

(e) $\sup_i E\left[\left(\frac{1}{T\sqrt{T}}\sum_{t=1}^T y_{it-1}\right)^4\right] < M$.

(f) $\sup_i E\left[\left(\frac{1}{T\sqrt{T}}\sum_{t=1}^T \frac{t-1}{T} y_{it-1}\right)^4\right] < M$.

Proof. The lemma follows by direct calculations and we omit the proof. ■

Lemma 14 *Suppose that Assumptions 1 – 3, and 6 hold. Then, the following hold.*

(a) $\sum_{i=1}^n \left(\hat{\sigma}_{1,iT}^2 - \sigma_i^2\right)^2 = o_p(1)$.

(b) $\sup_{1\le i\le n} \left(\hat{\sigma}_{1,iT}^2 - \sigma_i^2\right) = o_p(1)$.

(c) $\sum_{i=1}^n \left(\hat{\sigma}_{3,iT}^2 - \sigma_i^2\right)^2 = o_p(1)$.

(d) $\sup_{1\le i\le n} \left(\hat{\sigma}_{3,iT}^2 - \sigma_i^2\right) = o_p(1)$.

(e) *With probability approaching one, there exists a constant $M > 0$ such that $\inf_i \hat{\sigma}_{3,iT}^2 \ge M$.*

Proof.

Part (a): The required result follows by Lemma 5(a) if $\sum_{i=1}^n \left(\hat{\sigma}_{1,iT}^2 - \tilde{\sigma}_{iT}^2\right)^2 = o_p(1)$. Under Assumption 6, we have

$$\hat{\sigma}_{1,iT}^2 - \tilde{\sigma}_{iT}^2 = \frac{y_{i0}^2}{T} + \frac{\theta_i^2}{n^{1/2}T}\left(\frac{1}{T^2}\sum_{t=1}^T y_{it-1}^2\right) - 2\frac{\theta_i}{n^{1/4}T}\left(\frac{1}{T}\sum_{t=1}^T u_{it}y_{it-1}\right).$$

Then, by Lemma 13 and since the support of θ_i is uniformly bounded, we have for some constant M,

$$
\begin{aligned}
& \sum_{i=1}^{n} E\left(\hat{\sigma}_{1,iT}^{2}-\tilde{\sigma}_{iT}^{2}\right)^{2} \\
\leq \; & M\left[\frac{1}{T^{2}}\sum_{i=1}^{n}E\left(y_{i0}^{4}\right)+\frac{1}{nT^{2}}\sum_{i=1}^{n}E\left(\frac{1}{T^{2}}\sum_{t=1}^{T}y_{it-1}^{2}\right)^{2}+\frac{1}{n^{1/2}T^{2}}\sum_{i=1}^{n}\left(\frac{1}{T}\sum_{t=1}^{T}u_{it}y_{it-1}\right)^{2}\right] \\
= \; & O\left(\frac{n}{T^{2}}\right)+O\left(\frac{1}{T^{2}}\right)+O\left(\frac{n^{1/2}}{T^{2}}\right)=o\left(1\right), \qquad (9)
\end{aligned}
$$

as required. ■

Part (b): By the triangle inequality and Lemma 5(b), the required result follows if we show

$$\sup_{1\leq i\leq n}\left|\hat{\sigma}_{1,iT}^{2}-\tilde{\sigma}_{iT}^{2}\right|=o_{p}\left(1\right).$$

Since for any $\varepsilon>0$,

$$P\left\{\sup_{1\leq i\leq n}\left|\hat{\sigma}_{1,iT}^{2}-\tilde{\sigma}_{iT}^{2}\right|>\varepsilon\right\}\leq\sum_{i=1}^{n}P\left\{\left|\hat{\sigma}_{1,iT}^{2}-\tilde{\sigma}_{iT}^{2}\right|>\varepsilon\right\}\leq\frac{1}{\varepsilon^{2}}\sum_{i=1}^{n}E\left[\left(\hat{\sigma}_{1,iT}^{2}-\tilde{\sigma}_{iT}^{2}\right)^{2}\right]\rightarrow0$$

by (9), we have the required result. ■

Part (c): The required result follows by Part (a) and the inequality $(a+b)^2 \leq 2a^2+2b^2$, if we show that

$$\sum_{i=1}^{n}\left(\hat{\sigma}_{3,iT}^{2}-\hat{\sigma}_{1,iT}^{2}\right)^{2}=o_{p}\left(1\right),$$

for which it is enough by the Markov inequality to show that

$$\sum_{i=1}^{n}E\left(\hat{\sigma}_{3,iT}^{2}-\hat{\sigma}_{1,iT}^{2}\right)^{2}\rightarrow0. \qquad (10)$$

Notice by definition that

$$
\begin{aligned}
\hat{\sigma}_{3,iT}^{2} &= \hat{\sigma}_{1,iT}^{2}-\frac{1}{T}\left(\Delta\underline{Y}_{i}\right)'\Delta G\left(\Delta G'\Delta G\right)^{-1}\Delta G'\left(\Delta\underline{Y}_{i}\right) \\
&= \hat{\sigma}_{1,iT}^{2}-\frac{1}{T}\left(y_{i0}^{2}+\left(\frac{y_{iT}-y_{i0}}{\sqrt{T}}\right)^{2}\right).
\end{aligned}
$$

Then the required result (10) follows by Lemma 13(a) and (b). ■

Part (d): The required result follows by Part (b) and the triangle inequality, if we show

$$\sup_{1\leq i\leq n}\left|\hat{\sigma}_{3,iT}^{2}-\hat{\sigma}_{1,iT}^{2}\right|=o_{p}\left(1\right),$$

which follows since $\sup_{1\le i\le n}\left|\hat{\sigma}^2_{3,iT}-\hat{\sigma}^2_{1,iT}\right|^2 \le \sum_{i=1}^n\left(\hat{\sigma}^2_{3,iT}-\hat{\sigma}^2_{1,iT}\right)^2$ and by (10). ■

Part (e): Since $\inf_{1\le i\le n}\hat{\sigma}^2_{3,iT} \le \inf_i \sigma^2_i - \sup_{1\le i\le n}\left|\hat{\sigma}^2_{3,iT}-\sigma^2_i\right|$ under Assumption 1, the required result follows by Part (d). ■